International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 07
June 2017

# A STUDY ON SINGULAR POINT/SINGULARITY OF COMPLEX ANALYSIS

**Ali Ibrahim Mohammed**
**Master of Science (Mathematics)**
**University College of Science, Osmania University**

**Prof.B.Shankar**
**Prof. Dept. Of Mathematics**
**University College of Science, Osmania University**

**ABSTRACT:**

To study the Points at which a function *f*(z) is not analytic are called singular points or singularities of *f*(z). To study the different types of singularity of a complex function *f*(z) are discussed and the definition of a residue at a pole is given. The residue theorem is used to evaluate contour integrals where the only singularities of *f*(z) inside the contour are poles. The function f(z) has a singularity at $z = z_0$ and in a neighbourhood of $z_0$ (i.e. a region of the complex plane which contains $z_0$) there are no other singularities then $z_0$ is an isolated singularity of *f*(z).

To analyse the singularities are extremely important in complex analysis, where they characterize the possible behaviors of analytic functions. Complex singularities are points $z_0$ in the domain of a function $f$ where $f$ fails to be analytic. To analyze Isolated singularities may be classified as poles, essential singularities, logarithmic singularities, or removable singularities. Non-isolated singularities may arise as natural boundaries or branch cuts.

**INTRODUCTION:**

Singularity, also called singular point, of a function of the complex variable *z* is a point at which it is not analytic (that is, the function cannot be expressed as an infinite series in powers of *z*) although, at points arbitrarily close to the singularity, the function may be analytic, in which case it is called an isolated singularity. In general, because a function behaves in an anomalous manner at singular points, singularities must be treated separately when analyzing the function, or mathematical model, in which they appear.

For example, the function $f(z) = e^z/z$ is analytic throughout the complex plane—for all values of *z*—except at the point $z = 0$, where the series expansion is not defined because it contains the term $1/z$. The series is $1/z + 1 + z/2 + z^2/6 + \cdots + z^n/(n+1)! + \cdots$ where the factorial symbol ($k!$) indicates the product of the integers from *k* down to 1. When the function is bounded in a neighbourhood around a singularity, the function can be

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 07
June 2017

redefined at the point to remove it; hence it is known as a removable singularity. In contrast, the above function tends to infinity as $z$ approaches 0; thus, it is not bounded and the singularity is not removable (in this case, it is known as a simple pole). A **singularity** is in general a point at which a given mathematical object is not defined, or a point of an exceptional set where it fails to be well-behaved in some particular way, such as differentiability. See Singularity theory for general discussion of the geometric theory, which only covers some aspects.

For example, the function

$f(x) = 1/x$

On the real line has a singularity at $x = 0$, where it seems to "explode" to $\pm\infty$ and is not defined. The function $g(x) = |x|$ (see absolute value) also has a singularity at $x = 0$, since it is not differentiable there. Similarly, the graph defined by $y^2 = x$ also has a singularity at (0,0), this time because it has a "corner" (vertical tangent) at that point.

The algebraic set defined by $\{(x,y):|x|=|y|\}$in the $(x, y)$ coordinate system has a singularity (singular point) at (0, 0) because it does not admit a tangent there. Here  has singularity at z=0

; ; Rez; Im z; zRe z are nowhere analytic. That does not mean that every point of C is a singularity.

**Analytic Functions:**

In mathematics, an analytic function is a function that is locally given by a convergent power series. There exist both real analytic functions and complex analytic functions, categories that are similar in some ways, but different in others. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not hold generally for real analytic functions. A function is analytic if and only if its Taylor series about x0 converges to the function in some neighborhood for every x0 in its domain.

"Analytic functions" are functions that are locally representable as a power series. They behave quite differently in the real and complex case. The answer given by Anon describes the real case, but the modified zeta function you mention in your answer is actually a complex function.

Complex analytic functions are the heart and soul of complex analysis. They are defined, quite simply, as complex functions of a complex variable which possess a derivative, at least in a certain region of the complex plane. The incredible insight is that "having a derivative" in the complex context is a rare and unique property with far-reaching consequences. There's a lot you can say about a function when you know it's analytic, since there are "relatively few" analytic functions. They are very special.

For example, if you know the value of an analytic function along any closed curve, you also know its value at any point inside the area enclosed by the curve. This should seem quite shocking: if you consider the real case, for example a real-valued function of two real variables, there's absolutely nothing you can say about the values inside a region given the values on the boundary of that region - not even if the function is continuous, differentiable once or differentiable a million times. Functions that pop up "naturally" in the course of mathematical investigations, for example in number theory, are sometimes analytic in a certain region, but not necessarily throughout the complex plane. In those cases, it was discovered that it is immensely fruitful to "complete" the function by piecing together extensions of it that are defined, and are analytic, in larger and larger regions.

This again may seem strange: if I give you a very smooth and well-behaved real function of a real variable define in some interval, there's any number of ways to "extend it" to a still-smooth function on a larger interval. But analytic functions, you see, are extremely rigid and special; if there's any way at all to push the boundary of definition beyond the original domain, there's just one way of doing it. Number theorists like to encode number-theoretic data in a single function, such as a generating function or a Dirichlet series. This is a very clever idea: you take infinitely many numbers that have some useful meaning (like primes, or squares, or the number of objects of certain kind) and pack them into a single function. It then seems reasonable to believe that understanding the function will teach us something new about our original problem.

This happened many times, and is believed to happen in many cases where it's not yet actually proven. Few things excite a number theorist more that finding an analytic continuation, and a corresponding functional equation, for the function they are investigating. In a way, enormous parts of modern number theory can be seen as instances of this general idea. Riemann's original memoir on prime numbers and the zeta function was of this nature, and it opened up the entire world of analyti number theory. Subsequent discoveries by Dirichlet, Hasse, Hecke, Weil and many others pushed those ideas further and further into some of the deepest and most beautiful areas of modem mathematics.

**Definition 1.1 (Analytic Function).** The complex function $f(z)$ is analytic at the point $z_0$ provided there is some $\epsilon > 0$ such that $f'(z)$ exists for all $z \in D_\epsilon(z_0)$. In other words, $f(z)$ must be differentiable not only at $z_0$, but also at all points in some $\epsilon$-neighborhood of $z_0$.

If $f(z)$ is analytic at each point in the region $R$, then we say that $f(z)$ is an analytic function on $R$. Again, we have a special term if $f(z)$ is analytic on the whole complex plane.

**Definition 1.2 (Entire Function).** If $f(z)$ is analytic on the whole complex plane then $f(z)$ is said to be an entire function. Points of non-analyticity for a function are called singular points. They are important for applications in physics and engineering. Our definition of the derivative for complex functions is formally the same as for real functions and is the natural extension from real variables to complex variables. The basic differentiation formulas are identical to those for real functions, and we obtain the same rules for differentiating powers, sums, products, quotients, and compositions of functions. We can easily establish the proof of the differentiation formulas by using the limit theorems.

**Theorem 1.1.** If $f(z)$ is differentiable at $z_0$ then $f(z)$ is continuous at $z_0$.

**Proof:** From Equation (3-1), we obtain

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=f'(z_0).$$

Using the multiplicative property of limits given in Theorem 1.3, we get

$$\lim_{z\to z_0}(f(z)-f(z_0)) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}(z-z_0)$$

$$= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\lim_{z\to z_0}(z-z_0)$$

$$= f'(z_0)\times 0$$

$$= 0$$

This result implies that $\lim_{z\to z_0} f(z) - \lim_{z\to z_0} f(z_0) = 0$, which in turn implies that

$$\lim_{z\to z_0} f(z) = f(z_0).$$

Therefore, $f(z)$ is continuous at $z_0$.

**Proof.**

**The Derivative of** $f(z)g(z)$

International Journal of Research
Available at https://edupediapublications.org/journals
p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 07
June 2017

We can establish Equation (1-8) $\frac{d}{dz}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z)$ from Theorem 1.1.

Letting $h(z) = f(z)g(z)$ and using Definition 1.1, we write

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0}.$$

If we subtract and add the term $f(z_0)g(z)$ in the numerator, we get

$$h'(z_0) = \lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z) + f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z)}{z - z_0} + \lim_{z \to z_0} \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{(f(z) - f(z_0))g(z)}{z - z_0} + \lim_{z \to z_0} \frac{f(z_0)(g(z) - f(z_0)g(z_0))}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} g(z) + \lim_{z \to z_0} f(z_0) \lim_{z \to z_0} \frac{g(z) - f(z_0)g(z_0)}{z - z_0}$$

Using the definition of the derivative given by Equation (3-1) and the continuity of $g(z)$, we obtain

$$h'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0),$$

which is what we wanted to establish. We leave the proofs of the other differentiation rules as exercises. The rule for differentiating polynomials carries over to the complex case as well.

If we let $P(z)$ be a polynomial of degree $n$, so that

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n,$$

then mathematical induction, along with Equations (3-5) and (3-7), gives

$$P'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots + a_{n-1} z^{n-1}.$$

Again, we leave the details of this proof for the reader to finish, as an exercise. We shall use the differentiation rules as aids in determining when functions are analytic. For example, Equation (1-9) tells us that if $P(z)$ and $Q(z)$ are polynomials, then their quotient $\frac{P(z)}{Q(z)}$ is analytic at all points where $Q(z) \neq 0$. This condition implies that the function $f(z) = \frac{1}{z}$ is analytic for all $z \neq 0$. The square root function is more complicated. If $f(z) = z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{i\frac{Arg(z)}{2}}$, then $f(z)$ is analytic at all points except $z = 0$ (because $Arg(0)$ is undefined) and at points that lie along the negative $x$-

axis. Therefore the function $f(z) = z^{\frac{1}{2}}$, is not continuous at points that lie along the negative $x$-axis.

**SINGULARITIES, ZEROS, AND POLES:**

Recall that the point $z = \alpha$ is called a singular point, or singularity of the complex function f(z) if f is not analytic at $z = \alpha$, but every neighborhood $D_R(\alpha)$ of $\alpha$ contains at least one point at which f(z) is analytic. For example, the function $f(z) = \frac{1}{1-z}$ is not analytic at $z = 1$, but is analytic for all other values of z. Thus the point $z = 1$ is a singular point of f(z). As another example, consider $g(z) = \mathrm{Log}(z)$. We saw in Section 5.2 that g(z) is analytic for all z except at the origin and at all points on the negative real-axis. Thus, the origin and each point on the negative real axis is a singularity of $g(z) = \mathrm{Log}(z)$.

The point $\alpha$ is called a isolated singularity of the complex function f(z) if f is not analytic at $z = \alpha$, but there exists a real number $R > 0$ such that f(z) is analytic everywhere in the punctured disk $D_R^*(\alpha)$. The function $f(z) = \frac{1}{1-z}$ has an isolated singularity at $z = 1$.

The function $g(z) = \mathrm{Log}(z)$, however, the singularity at $z = 0$ (or at any point of the negative real axis) that is not isolated, because any neighborhood of contains points on the negative real axis, and $g(z) = \mathrm{Log}(z)$ is not analytic at those points. Functions with isolated singularities have a Laurent series because the punctured disk $D_R^*(\alpha)$ is the same as the annulus $A(\alpha, 0, R)$. The logarithm function $g(z) = \mathrm{Log}(z)$ does not have a Laurent series at any point $z = -a$ on the negative real-axis. We now look at this special case of Laurent's theorem in order to classify three types of isolated singularities.

**Definition (Removable Singularity, Pole of order k, Essential Singularity).** Let f(z) have an isolated singularity at $\alpha$ with Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

Then we distinguish the following types of singularities at $\alpha$.

(i) If $c_n = 0$ for $n = -1, -2, -3, \ldots$, then we say that f(z) has a removable singularity at $\alpha$.

(ii) If k is a positive integer such that $c_{-k} \neq 0$ but $c_n = 0$ for $n = -k-1, -k-2, -k-3, \ldots$, then we say that f(z) has a pole of order k at $\alpha$

(iii) If $c_n \neq 0$ for infinitely many negative integers n, then we say that f(z) has an essential singularity at $z = \alpha$.

Let's investigate some examples of these three cases.

(i). If f(z) has a removable singularity at $z = \alpha$, then it has a Laurent series $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ valid for $z \in A(\alpha, 0, R)$.

Theorem 4.17 (see Section 4.4) implies that the power series for f(z) defines an analytic function in the disk $D_R(\alpha)$.

If we use this series to define $f(\alpha) = c_0$, then the function f(z) becomes analytic at $z = \alpha$, removing the singularity.

For example, consider the function $f(z) = \frac{\sin(z)}{z}$. It is undefined at $z = 0$ and has an isolated singularity at $z = 0$, as the Laurent series for f(z) is

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \ldots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \frac{z^{10}}{11!} + \ldots$$

valid for $|z| > 0$.

We can remove this singularity if we define $f(0) = 1$, for then f(z) will be analytic at $z = 0$ in accordance. Another example is $g(z) = \frac{\cos(z) - 1}{z^2}$, which has an isolated singularity at the point $z = 0$, as the Laurent series for g(z) is

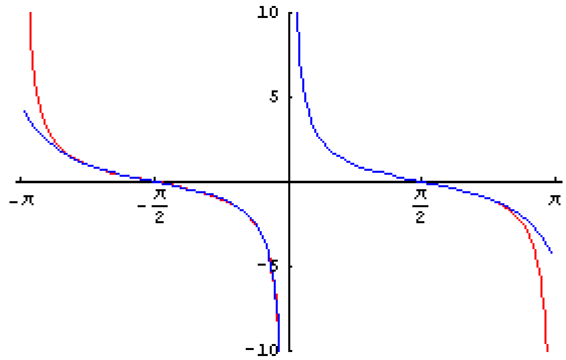
$$g(z) = \frac{1}{z^2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots\right)$$

$$= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \frac{z^6}{8!} - \frac{z^8}{10!} + \ldots$$

valid for $|z| > 0$. If we define $f(0) = -\frac{1}{2}$, then g(z) will be analytic for all z.

(ii). If f(z) has a pole of order k at $z = \alpha$, the Laurent series for f(z) is $f(z) = \sum_{n=-k}^{\infty} c_n (z-\alpha)^n$ valid for $z \in A(\alpha, 0, R)$. where $c_{-k} \neq 0$.

**Extra Example:** The following example will help this concept. Consider the function $f(z) = \cot z$. The leading term in the Laurent series expansion S(z) is $\frac{1}{z}$ and S(z) goes to $\infty$ as $z \to 0$ in the same manner as $\cot z$.




If f(z) has a pole of order 1 at $z = \alpha$, we say that f(z) has a simple pole at $z = \alpha$.
For example,

$$g(z) = \frac{1}{z} e^z = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^{n-1}$$

$$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \frac{z^4}{5!} + \frac{z^5}{6!} + \ldots$$

has a simple pole at $z = 0$.

**Theorem 7.10** A function f analytic in $D_R(\alpha)$ has a zero of order k at the point α if *f* its Taylor series given by has $c_0 = c_1 = \ldots = c_{k-1} = 0$, but $c_k = 0$.

**Proof** The conclusion follows immediately from Definition 7.6, because we have

$c_n = f(n)(\alpha)/n!$ according to Taylor's theorem.

## APPLICATIONS OF RESIDUES

Recall the definition of improper integrals in calculus:

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 07
June 2017

$$\int_0^{\infty} f(x)dx = \lim_{R\to\infty} \int_0^{R} f(x)dx,$$
$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)dx.$$

The Cauchy Principal Value (P.V.) is given by

$$\text{P. V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$$

The Cauchy principal value of an improper integral is not necessarily the same as the improper integral. For example,

$$\text{P. V} \int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx = 0,$$

while

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} x\,dx + \lim_{R_2\to\infty} \int_0^{R_2} x\,dx = -\infty + \infty$$

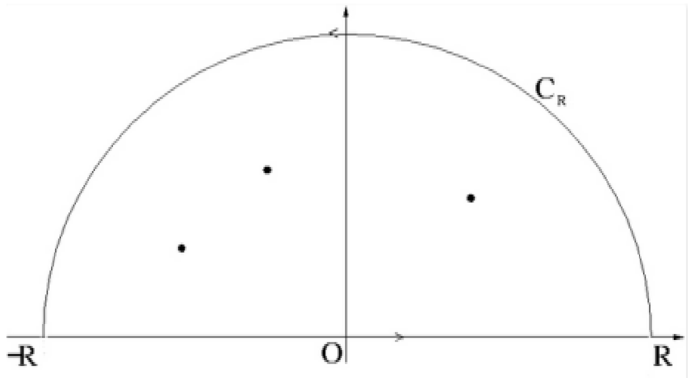
is undefined. In general, if $<\infty$ then R.V$<\infty$, but the converse need not be true. Suppose that f(x) is an even function. Then

$$\int_0^{R} f(x)dx = \frac{1}{2} \int_{-R}^{R} f(x)dx,$$
$$\int_{-R_1}^{0} f(x)dx = \int_0^{R_1} f(x)dx.$$

So,

$$\text{P. V.} \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx.$$

Let us consider an even function f(x) of the form f(x)=p(x)/q(x), where p(x), q(x) are polynomials with real coefficients no factors in common. Furthermore, we assume that q(z) has no real zeros but has at least one zero above the real axis. Let us consider a positively oriented upper semicircle $C_R$ whose radius R is large enough to contain all the zeros above the real axis as shown in the figure below.




$C_R$ together with the interval $[-R,R]$ form a positively oriented simple closed contour. Then by Cauchy's Residue Theorem we have

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z),$$

i.e.

$$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z) - \int_{C_R} f(z)dz.$$

If then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z).$$

If in addition $f$(x) is even,

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$

$$\int_{0}^{\infty} f(x)dx = \pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z).$$

**Theorem 3.1:**

A function $f(z)$ analytic in $D_R(\alpha)$ has a zero of order k at the point $z=\alpha$ iff its Taylor series given by $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has

$c_0 = c_1 = \dots = c_{k-1} = 0$ and $c_k \neq 0$.

The conclusion follows immediately from Definition 7.6, because we have

According to Taylor's theorem.

**Theorem 3.2:** Suppose f(z) is analytic in $D_R(\alpha)$ . Then f(z) has a zero of order k at the point $z = \alpha$ if and only if it can be expressed in the form

$f(z) = (z - \alpha)^k g(z)$ ,

Where g(z) is analytic at $z = \alpha$ and $g(\alpha) \neq 0$.

Proof Suppose that f has a zero of order k at the point α and that

Theorem 3.1 assures us that $c_n = 0$ for $0 \leq n \leq k - 1$ and that $c_k = 0$, so that we can write f as

where $c_k \neq 0$. The series on the right side of Equation (3.1) defines a function, which we denote by g. That is,

By Theorem 3.1, g is analytic in DR (α), and $g(\alpha) = c_k \neq 0$.

Conversely, suppose that f has the form given by Equation(3.1). Since g is analytic at α, it has the power series representation

where $g(\alpha) = b_0 \neq 0$ by assumption. If we multiply both sides of the expression defining g (z) by $(z - \alpha)^k$, we get

By Theorem 3.1, f has a zero of order k at the point α, and our proof is complete.

**Conclusion:**

The study concludes the complex function in singular point $z_0$ is called an isolated singular point of an analytic function *f*(z) if there exists a deleted ε-spherical neighborhood of $z_0$ that contains no singularity. If no such neighborhood can be found, $z_0$ is called a non-isolated singular point. Thus an isolated singular point is a singular point that stands completely by itself, embedded in regular points. Where $z_1$, $z_2$ and $z_3$ are

p-ISSN: 2348-6848
e-ISSN: 2348-795X
Volume 04 Issue 07
June 2017

isolated singular points. Most singular points are isolated singular points. A non-isolated singular point is a singular point such that every deleted ε-spherical neighborhood of it contains singular points. See Fig. 1b where $z_0$ is the limit point of a set of singular points. Isolated singular points include poles, removable singularities, essential singularities and branch points.

The study a analysis of analytic functions and complex analytic functions, categories that are similar in some ways, but different in others. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not hold generally for real analytic functions. The relationship of the residue theorem to Stokes' theorem is given by the Jordan curve theorem. evaluate the residue of a function $f(z)$ with one singular point in a contour using Laurent series expansion. However, how do we proceed when encloses more than one isolated singular points? In such a situation, we have to extend the concept of residue developed so far to more than one singularity. The theorem of residues deals with such a general case and we discuss it in the following section.

**Reference:**

1. Milnor, J. (2016). Singular Points of Complex Hypersurfaces.(AM-61) (Vol. 61). Princeton University Press.
2. Rudin, W. (1987). Real and complex analysis. Tata McGraw-Hill Education.
3. Campbell, S. L. V. (1982). Singular systems of differential equations II. Pitman Publishing (UK).
4. Pemantle, R., & Wilson, M. C. (2002). Asymptotics of multivariate sequences: I. smooth points of the singular variety. Journal of Combinatorial Theory, Series A, 97(1), 129-161.
5. Ahlfors, L. V. (1953). Complex analysis: an introduction to the theory of analytic functions of one complex variable. *New York, London*, 177.
6. Krantz, S. G. (1992). *Function theory of several complex variables*. Pacific Grove, California: Wadsworth & Brooks/Cole Advanced Books & Software.
7. Rudin, W. (1987). *Real and complex analysis*. Tata McGraw-Hill Education.
8. Hormander, L. (1973). *An introduction to complex analysis in several variables* (Vol. 7). Elsevier.
9. Gamelin, T. (2003). *Complex analysis*. Springer Science & Business Media.
10. Burckel, R. B. (1980). *An introduction to classical complex analysis* (Vol. 1). Academic Press.
11. Bak, J., & Newman, D. J. (2010). *Complex analysis*. Springer Science & Business Media.
12. Nevanlinna, R., & Paatero, V. (1969). Introduction to complex analysis.

13. Fisher, S. D. (2007). *Function theory on planar domains: a second course in complex analysis*. Courier Corporation.
14. Mathews, J. H., & Howell, R. W. (2012). *Complex analysis for mathematics and engineering*. Jones & Bartlett Publishers.
15. Krantz, S. G. (2004). *Complex analysis: the geometric viewpoint* (No. 23). MAA.
16. Narasimhan, R., & Nievergelt, Y. (2012). *Complex analysis in one variable*. Springer Science & Business Media.