Study and Analysis of Fourier Transformation

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ABSTRACT

This Paper has evolved from regularly teaching courses in integral transforms, boundary value problems, differential equations, applied mathematics, and advanced engineering mathematics over many years to students of mathematics and engineering Field . It is essentially designed to cover advanced mathematical methods for science and engineering students with heavy emphasis on many different and varied applications

INTRODUCTION

Born in France in 1768, Jean Baptiste Joseph, Baron de Fourier came up with this piece of maths in his thirties after puzzling over problems with agriculture. Of great interest to him was the fact that the surface of the ground got quite hot during the day and cold during the night, but deeper down the temperatures were more nearly constant. He was seeking to understand heat flow.Quarter of a millennium on, students of maths and engineering remain convinced he was seeking to impose misery and gloom on future generations. You will see that fourier analysis is used in almost every aspect of the subject ranging from solving linear differential equations to developing computer models, to the processing and analysis of data.

PHYSICAL SIGNIFICANCE OF FOURIER TRANSFORM

The Fourier Transform converts a set of time domain data vectors into a set of frequency (or per time) domain vectors.Imagine you wanted to know about changes in soil temperature . Now suppose you to measure the temperature of the soil in your garden at dawn, midday, dusk and midnight, every day for a year. You would then have a list of real numbers representing the soil temperatures.Now if we plot these readings on a graph the vertical y axis would be labelled 'temperature' and the horizontal x axis would be labelled ' time', we get a so called 'time domain' graph.From the graph we infer the nights are cold and the days are warm and as it seems obvious, summer is warmer than winter!The graph you've just imagined is (roughly) the sum of two sinusoids (sine waves). One with a frequency of one day as the temperature varies between day and night, the other with a frequency of one year as the temperature varies with the seasons.The Fourier Transform provides a means of manipulating - or transforming - this raw data into an alternative set of data, the magnitude of which which can be plotted on a graph with

differently labelled axis. Never mind the y axis label for now, but the x axis would now be labelled 'frequency' or 'per time' - this is in the 'frequency domain' graph.This graph looks very different! It will consist of two vertical lines rising from the frequency axis, one at a frequency (or period) of one day, the other at a frequency (or period) of one year. Thus we have 'analysed' a seemingly complicated (rather than complex in the mathematical sense) set of data and extracted the most interesting facts from it - days are warmer than nights, and summer is warmer than winter! There is still more information we can extract from the raw results of the Transform not shown on such a graph, we can look at the 'angle' information of the raw results which might tell us that the coldest day isn't January 1st, and midnight isn't the coldest part of the night.We could use exactly the same Fourier Transform for more interesting purposes - we could for example sample some music, transform it, and plot the 'frequency spectrum' to reproduce those dancing bars of LEDs on the stereo system!
The Fourier transform is a generalization of the complex Fourier series in the limit as $L \to \infty$. Replace the discrete A_n with the continuous $F(k)\,dk$ while letting $n/L \to k$ Then change the sum to an integral, and the equations become

$$f(x) = \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk \tag{1}$$
$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx. \tag{2}$$

Here,

$$F(k) = \mathcal{F}_x[f(x)](k) \tag{3}$$
$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx \tag{4}$$

is called the *forward* $(-i)$ Fourier transform, and

$$f(x) = \mathcal{F}_k^{-1}[F(k)](x) \tag{5}$$
$$= \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk \tag{6}$$

is called the *inverse* $(+i)$ Fourier transform. The notation $\mathcal{F}_x[f(x)](k)$ is introduced in Trott (2004, p. xxxiv), and $\hat{f}(k)$ and $\check{f}(x)$ are sometimes also used to denote the Fourier transform and inverse Fourier transform, respectively (Krantz 1999, p. 202).

Note that some authors (especially physicists) prefer to write the transform in terms of angular frequency $\omega \equiv 2\pi\nu$ instead of the oscillation frequency ν. However, this destroys the symmetry, resulting in the transform pair

$$H(\omega) = \mathcal{F}[h(t)] \tag{7}$$
$$= \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt \tag{8}$$
$$h(t) = \mathcal{F}^{-1}[H(\omega)] \tag{9}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega. \tag{10}$$

To restore the symmetry of the transforms, the convention

$$g(y) = \mathcal{F}[f(t)] \tag{11}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iyt} dt \tag{12}$$
$$f(t) = \mathcal{F}^{-1}[g(y)] \tag{13}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{iyt} dy \tag{14}$$

is sometimes used (Mathews and Walker 1970, p. 102).

In general, the Fourier transform pair may be defined using two arbitrary constants a and b as

$$F(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(t) e^{ib\omega t} dt \tag{15}$$
$$f(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} F(\omega) e^{-ib\omega t} d\omega. \tag{16}$$

The Fourier transform $F(k)$ of a function $f(x)$ is implemented the Wolfram Language as FourierTransform[f, x, k], and different choices of a and b can be used by passing the optional FourierParameters-> $\{a, b\}$ option. By default, the Wolfram Language takes FourierParameters as $(0, 1)$ Unfortunately, a number of other conventions are in widespread use. For example, $(0, 1)$ is used in modern physics, $(1, -1)$ is used in pure mathematics and systems engineering, $(1, 1)$ is used in probability theory for the computation of the characteristic function, $(-1, 1)$ is used in classical physics, and $(0, -2\pi)$ is used in signal processing. In this work, following Bracewell (1999, pp. 6-7), *it is always assumed that* $a = 0$ *and* $b = -2\pi$ *unless otherwise stated.* This choice often results in greatly simplified transforms of common functions such as 1, $\cos(2\pi k_0 x)$, etc.

Since any function can be split up into even and odd portions $E(x)$ and $O(x)$,

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)] \tag{17}$$
$$= E(x) + O(x), \tag{18}$$

a Fourier transform can always be expressed in terms of the Fourier cosine transform and Fourier sine transform as

$$\mathcal{F}_x[f(x)](k) = \int_{-\infty}^{\infty} E(x)\cos(2\pi k x)\,dx - i\int_{-\infty}^{\infty} O(x)\sin(2\pi k x)\,dx. \tag{19}$$

A function $f(x)$ has a forward and inverse Fourier transform such that

$$f(x) = \begin{cases} \int_{-\infty}^{\infty} e^{2\pi i k x} \left[\int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx \right] dk & \text{for } f(x) \text{ continuous at } x \\ \frac{1}{2} [f(x_+) + f(x_-)] & \text{for } f(x) \text{ discontinuous at } x, \end{cases} \tag{20}$$

provided that

1. $\int_{-\infty}^{\infty} |f(x)| \, dx$ exists.

2. There are a finite number of discontinuities.

3. The function has bounded variation. A sufficient weaker condition is fulfillment of the Lipschitz condition

(Ramirez 1985, p. 29). The smoother a function (i.e., the larger the number of continuous derivatives), the more compact its Fourier transform.

The Fourier transform is linear, since if $f(x)$ and $g(x)$ have Fourier transforms $F(k)$ and $G(k)$, then

$$\int [a f(x) + b g(x)] e^{-2\pi i k x} \, dx = a \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx + b \int_{-\infty}^{\infty} g(x) e^{-2\pi i k x} \, dx \tag{21}$$

$$= a F(k) + b G(k). \tag{22}$$

Therefore,

$$\mathcal{F}[a f(x) + b g(x)] = a \mathcal{F}[f(x)] + b \mathcal{F}[g(x)] \tag{23}$$

$$= a F(k) + b G(k). \tag{24}$$

The Fourier transform is also symmetric since $F(k) = \mathcal{F}_x[f(x)](k)$ implies $F(-k) = \mathcal{F}_x[f(-x)](k)$.

Let $f * g$ denote the convolution, then the transforms of convolutions of functions have particularly nice transforms,

$$\mathcal{F}[f * g] = \mathcal{F}[f] \mathcal{F}[g] \tag{25}$$

$$\mathcal{F}[f g] = \mathcal{F}[f] * \mathcal{F}[g] \tag{26}$$

$$\mathcal{F}^{-1}[\mathcal{F}(f) \mathcal{F}(g)] = f * g \tag{27}$$

$$\mathcal{F}^{-1}[\mathcal{F}(f) * \mathcal{F}(g)] = f g. \tag{28}$$

The first of these is derived as follows:

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i k x} f(x') g(x - x') \, dx' \, dx \tag{29}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[e^{-2\pi i k x'} f(x') \, d x' \right] \left[e^{-2\pi i k (x - x')} g(x - x') \, d x \right] \tag{30}$$

$$= \left[\int_{-\infty}^{\infty} e^{-2\pi i k x'} f(x') \, d x' \right] \left[\int_{-\infty}^{\infty} e^{-2\pi i k x''} g(x'') \, d x'' \right] \tag{31}$$

$$= \mathcal{F}[f] \mathcal{F}[g], \tag{32}$$

where $x'' \equiv x - x'$.

There is also a somewhat surprising and extremely important relationship between the autocorrelation and the Fourier transform known as the Wiener-Khinchin theorem. Let $\mathcal{F}_x[f(x)](k) = F(k)$, and $\bar{f}$ denote the complex conjugate of f, then the Fourier transform of the absolute square of $F(k)$ is given by

$$\mathcal{F}_k\left[|F(k)|^2\right](x) = \int_{-\infty}^{\infty} \bar{f}(\tau) f(\tau + x) \, d\tau. \tag{33}$$

The Fourier transform of a derivative $f'(x)$ of a function $f(x)$ is simply related to the transform of the function $f(x)$ itself. Consider

$$\mathcal{F}_x[f'(x)](k) = \int_{-\infty}^{\infty} f'(x) e^{-2\pi i k x} \, d x. \tag{34}$$

Now use integration by parts

$$\int v \, d u = [u v] - \int u \, d v \tag{35}$$

with

$$d u = f'(x) \, d x \tag{36}$$

$$v = e^{-2\pi i k x} \tag{37}$$

and

$$u = f(x) \tag{38}$$

$$d v = -2\pi i k e^{-2\pi i k x} \, d x, \tag{39}$$

then

$$\mathcal{F}_x[f'(x)](k) = \left[f(x) e^{-2\pi i k x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \left(-2\pi i k e^{-2\pi i k x} \, d x \right). \tag{40}$$

The first term consists of an oscillating function times $f(x)$. But if the function is bounded so that

$$\lim_{x \to \pm\infty} f(x) = 0 \tag{41}$$

(as any physically significant signal must be), then the term vanishes, leaving

$$\mathcal{F}_x[f'(x)](k) = 2\pi i k \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx \tag{42}$$
$$= 2\pi i k \mathcal{F}_x[f(x)](k). \tag{43}$$

This process can be iterated for the nth derivative to yield

$$\mathcal{F}_x\left[f^{(n)}(x)\right](k) = (2\pi i k)^n \mathcal{F}_x[f(x)](k). \tag{44}$$

The important modulation theorem of Fourier transforms allows $\mathcal{F}_x[\cos(2\pi k_0 x) f(x)](k)$ to be expressed in terms of $\mathcal{F}_x[f(x)](k) = F(k)$ as follows,

$$\mathcal{F}_x[\cos(2\pi k_0 x) f(x)](k) \equiv \int_{-\infty}^{\infty} f(x) \cos(2\pi k_0 x)\, e^{-2\pi i k x}\, dx \tag{45}$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{2\pi i k_0 x}\, e^{-2\pi i k x}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k_0 x}\, e^{-2\pi i k x}\, dx \tag{46}$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i (k-k_0) x}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i (k+k_0) x}\, dx \tag{47}$$
$$= \frac{1}{2}[F(k-k_0) + F(k+k_0)]. \tag{48}$$

Since the derivative of the Fourier transform is given by

$$F'(k) \equiv \frac{d}{dk}\mathcal{F}_x[f(x)](k) = \int_{-\infty}^{\infty} (-2\pi i x) f(x)\, e^{-2\pi i k x}\, dx, \tag{49}$$

it follows that

$$F'(0) = -2\pi i \int_{-\infty}^{\infty} x f(x)\, dx. \tag{50}$$

Iterating gives the general formula

$$\mu_n \equiv \int_{-\infty}^{\infty} x^n f(x)\, dx \tag{51}$$
$$= \frac{F^{(n)}(0)}{(-2\pi i)^n}. \tag{52}$$

The variance of a Fourier transform is

$$\sigma_f^2 = \left\langle (x f - \langle x f \rangle)^2 \right\rangle, \tag{53}$$

and it is true that

$$\sigma_{f+g} = \sigma_f + \sigma_g. \tag{54}$$

If $f(x)$ has the Fourier transform $\mathcal{F}_x[f(x)](k)=F(k)$, then the Fourier transform has the shift property

$$\int_{-\infty}^{\infty} f(x-x_0)\, e^{-2\pi i k x}\, dx = \int_{-\infty}^{\infty} f(x-x_0)\, e^{-2\pi i (x-x_0) k}\, e^{-2\pi i (k x_0)}\, d(x-x_0) \tag{55}$$

$$= e^{-2\pi i k x_0}\, F(k), \tag{56}$$

so $f(x-x_0)$ has the Fourier transform

$$\mathcal{F}_x[f(x-x_0)](k) = e^{-2\pi i k x_0}\, F(k). \tag{57}$$

If $f(x)$ has a Fourier transform $\mathcal{F}_x[f(x)](k)=F(k)$, then the Fourier transform obeys a similarity theorem.

$$\int_{-\infty}^{\infty} f(a x)\, e^{-2\pi i k x}\, dx = \frac{1}{|a|}\int_{-\infty}^{\infty} f(a x)\, e^{-2\pi i (a x)(k/a)}\, d(a x) = \frac{1}{|a|} F\left(\frac{k}{a}\right), \tag{58}$$

so $f(a x)$ has the Fourier transform

$$\mathcal{F}_x[f(a x)](k) = |a|^{-1} F\left(\frac{k}{a}\right). \tag{59}$$

The "equivalent width" of a Fourier transform is

$$w_e \equiv \frac{\int_{-\infty}^{\infty} f(x)\, dx}{f(0)} \tag{60}$$

$$= \frac{F(0)}{\int_{-\infty}^{\infty} F(k)\, dk}. \tag{61}$$

The "autocorrelation width" is

$$w_a \equiv \frac{\int_{-\infty}^{\infty} f \star \bar{f}\, dx}{\left[f \star \bar{f}\right]_0} \tag{62}$$

$$= \frac{\int_{-\infty}^{\infty} f\, dx \int_{-\infty}^{\infty} \bar{f}\, dx}{\int_{-\infty}^{\infty} f\, \bar{f}\, dx}, \tag{63}$$

where $f \star g$ denotes the cross-correlation of f and g and $\bar{f}$ is the complex conjugate.

Any operation on $f(x)$ which leaves its area unchanged leaves $F(0)$ unchanged, since

$$\int_{-\infty}^{\infty} f(x)\, dx = \mathcal{F}_x[f(x)](0) = F(0). \tag{64}$$

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The following table summarized some common Fourier transform pairs.

function	$f(x)$	$F(k) = \mathcal{F}_x[f(x)](k)$
Fourier transform--1	1	$\delta(k)$
Fourier transform--cosine	$\cos(2\pi k_0 x)$	$\frac{1}{2}[\delta(k-k_0)+\delta(k+k_0)]$
Fourier transform--delta function	$\delta(x-x_0)$	$e^{-2\pi i k x_0}$
Fourier transform--exponential function	$e^{-2\pi k_0 \lvert x \rvert}$	$\frac{1}{\pi}\frac{k_0}{k^2+k_0^2}$
Fourier transform--Gaussian	$e^{-a x^2}$	$\sqrt{\frac{\pi}{a}}\, e^{-\pi^2 k^2/a}$
Fourier transform--Heaviside step function	$H(x)$	$\frac{1}{2}\left[\delta(k)-\frac{i}{\pi k}\right]$
Fourier transform--inverse function	$-PV\frac{1}{\pi x}$	$i[1-2H(-k)]$
Fourier transform--Lorentzian function	$\frac{1}{\pi}\frac{\frac{1}{2}\Gamma}{(x-x_0)^2+\left(\frac{1}{2}\Gamma\right)^2}$	$e^{-2\pi i k x_0-\Gamma\pi\lvert k \rvert}$
Fourier transform--ramp function	$R(x)$	$\pi i \delta'(2\pi k)-\frac{1}{4\pi^2 k^2}$
Fourier transform--sine	$\sin(2\pi k_0 x)$	$\frac{1}{2} i[\delta(k+k_0)-\delta(k-k_0)]$

In two dimensions, the Fourier transform becomes

$$F(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x,k_y)\, e^{-2\pi i (k_x x + k_y y)}\, d k_x\, d k_y \tag{65}$$

$$f(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\, e^{2\pi i (k_x x + k_y y)}\, d x\, d y. \tag{66}$$

Similarly, the n-dimensional Fourier transform can be defined for $\mathbf{k}$, $\mathbf{x} \in \mathbb{R}^n$ by

$$F(\mathbf{x}) = \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} f(\mathbf{k})\, e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d^n\mathbf{k} \tag{67}$$

$$f(\mathbf{k}) = \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} F(\mathbf{x})\, e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d^n\mathbf{x}. \tag{68}$$

SEE ALSO

:Autocorrelation, Convolution, Discrete Fourier Transform, Fast Fourier Transform, Fourier Series, Fourier-Stieltjes Transform, Fourier Transform--1, Fourier Transform--Cosine, Fourier Transform--Delta Function, Fourier Transform--Exponential Function, Fourier Transform--Gaussian, Fourier Transform--Heaviside Step Function, Fourier Transform--Inverse Function, Fourier Transform--Lorentzian Function, Fourier Transform--Ramp Function, Fourier

Transform--Rectangle Function, Fractional Fourier Transform, Hankel Transform, Hartley Transform, Integral Transform, Laplace Transform, Parseval's Theorem, Structure Factor, Wiener-Khinchin Theorem, Winograd Transform

APPLICATIONS OF FOURIER TRANSFORM:

- Designing and using antennas
- Image Processing and filters
- Transformation, representation, and encoding
- Smoothing and sharpening
- Restoration, blur removal, and Wiener filter
- Data Processing and Analysis
- Seismic arrays and streamers
- Multibeam echo sounder and side scan sonar
- Interferometers — VLBI — GPS
- Synthetic Aperture Radar (SAR) and Interferometric SAR (InSAR)
- High-pass, low-pass, and band-pass filters
- Cross correlation — transfer functions — Coherence
- Signal and noise estimation — encoding time series.

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