

# Graphical Method to Solve Linear Programs

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## Abstract–

*A key problem faced by managers is how to allocate scarce resources among activities or projects. Linear programming, or LP, is a method of allocating resources in an optimal way. It is one of the most widely used operations research (OR) tools. It has been used successfully as a decision-making aid in almost all industries, and in financial and service organizations. Programming refers to mathematical programming. In this context, it refers to a planning process that allocates resources—labor, materials, machines, and capital—in the best possible (optimal) way so that costs are minimized or profits are maximized. In LP, these resources are known as decision variables. The criterion for selecting the best values of the decision variables (e.g., to maximize profits or minimize costs) is known as the objective function.*

## 1. Introduction

This publication will introduce a small LP problem that can be solved graphically. In other words, we'll plot the appropriate information on a graph, and then use the graph to find a solution to the problem.

In EM 8720, Using the Simplex Method to Solve Linear Programming Maximization Problems, we'll build on the graphical example and introduce an algebraic technique known as the simplex method. This method lets us solve very large LP problems [1] that would be impossible to solve graphically or without the analytical ability

of a computer. In other publications, we'll define several special types of LP and use forest products examples to help explain their functions.

For more complex problems, many software packages are available for obtaining a solution. It's just a matter of choosing the one you wish to work with. In a later publication, we'll use commercial software to solve larger LP problems and to examine how to use sensitivity analysis to gain more information than that provided by a simple optimal solution.

## 2. GRAPHICAL METHOD

### 2.1. Steps in solving an LP problem graphically

The steps in solving an LP problem graphically are introduced briefly below. In the remainder of this publication, we'll apply these steps to a simple LP problem.

### 2.2. Step 1. Formulate the LP problem.

Formulation refers to translating the real-world problem into a format of mathematical equations that represent the objective function and the constraint set. Often, data gathering, problem definition, and problem formulation are the most important (and the most difficult, time-consuming, and expensive) steps when using any OR tool.

A thorough understanding of the problem is necessary in order to formulate it correctly. During the formulation stage, an OR specialist may discover new insights into the problem that may change the scope of the original problem. For this reason, it's very important to obtain the help of those who work most closely with the system

being studied[2]. They can help the OR specialist collect the necessary data and correctly define the problem so that he or she can create a valid model. In formulating an LP problem, it often helps to put all of the relevant information into a table

**2.3. Step 2. Construct a graph and plot the constraint lines.**

Constraint lines represent the limitations on available resources. Usually, constraint lines are drawn by connecting the horizontal and vertical intercepts found from each constraint equation.

**2.4. Step 3. Determine the valid side of each constraint line.** The simplest way to start is to plug in the coordinates of the origin (0,0) and see whether this point satisfies the constraint. If it does, then all points on the origin side of the line are feasible (valid), and all points on the other side of the line are infeasible (invalid). If (0,0) does not satisfy the constraint, then all points on the other side and away from the origin are feasible (valid), and all points on the origin side of the constraint line are infeasible (invalid). There are two exceptions, which we will discuss later.

**2.5. Step 4. Identify the feasible solution region.** The feasible solution region represents the area on the graph that is valid for all constraints. Choosing any point in this area will result in a valid solution.

**2.6. Step 5. Plot two objective function lines to determine the direction of improvement.** Improvement is in the direction of greater value when the objective is to maximize the objective function, and is in the direction of lesser value when the objective is to minimize the objective function. The objective function lines do not have to include any of the feasible region [3] to determine the desirable direction to move.

**2.7. Step 6. Find the most attractive corner.** Optimal solutions always occur at corners. The most attractive corner is the last point in the feasible solution region touched by a line that is parallel to the two objective function lines drawn

in step 5 above. When more than one corner corresponds to an optimal solution, each corner and all points [4] along the line connecting the corners correspond to optimal solutions. We'll use an example to illustrate optimal solutions later.

**2.8. Step 7.** Determine the optimal solution by algebraically calculating coordinates of the most attractive corner.

**2.9. Step 8.** Determine the value of the objective function for the optimal solution.

**3. An example**

- An LP example: Production of woodentables and chairs*

Let's look at the profit maximization problem that our furniture manufacturer faces. The company uses wood and labor to produce tables and chairs. Recall that unit profit for tables is \$6, and unit profit for chairs is \$8. There are 300 board feet (bf) of wood available, and 110 hours of labor available. It takes 30 bf and 5 hours to make a table, and 20 bf and 10 hours to make a chair. Table 1 contains the information for the LP problem. We will go through the step-by-step process of solving this problem graphically.

*Table 1.—Information for the wooden tables and chairs linear programming problem.*

Resource	Table ( $X_1$ )	Chair ( $X_2$ )	Available
Wood (bf)	30	20	300
Labor (hr)	5	10	110
Unit profit	\$6	\$8	

**Step 1. Formulate the LP problem.**

Table 1 will help us formulate the problem. The bottom row is used to formulate the objective function. Objective functions are developed in such a way as to be either maximized or minimized. In this case, the company's management wishes to maximize unit profit[5]. The wood and labor rows are used to

formulate the constraint set. Finally, the nonnegativity conditions are stated.

Maximize:  $Z = 6X_1 + 8X_2$  (objective function)  
 Subject to:  $30X_1 + 20X_2 \leq 300$  (wood constraint: 300 bf available)  
 $5X_1 + 10X_2 \leq 110$  (labor constraint: 110 hours available)  
 $X_1, X_2 \geq 0$  (nonnegativity conditions)

Since only two variables (wood and labor) exist in this problem, it can be solved graphically. If there were more than two variables, the graph would have to be more than two dimensions.

**Step 2. Construct the graph and plot constraint lines.**

Draw the graph with the x axis representing the number of tables and the y axis representing the number of chairs. Plot the two constraint lines by finding the x and y intercepts for the two constraint equations in the following manner. First, rewrite the constraint inequalities as equalities and solve to obtain the intercepts:

Wood	Labor
$30X_1 + 20X_2 = 300$	$5X_1 + 10X_2 = 110$
Set $X_2 = 0$ and solve for $X_1$	Set $X_2 = 0$ and solve for $X_1$
$30X_1 = 300$	$5X_1 = 110$
$X_1 = 300/30$	$X_1 = 110/5$
= 10 tables	= 22 tables
<i>(All the wood is used to make tables.)</i>	<i>(All the labor is used to make tables.)</i>

Next:

Set $X_1 = 0$ and solve for $X_2$	Set $X_1 = 0$ and solve for $X_2$
$20X_2 = 300$	$10X_2 = 110$
$X_2 = 300/20$	$X_2 = 110/10$
= 15 chairs	= 11 chairs
<i>(All the wood is used to make chairs.)</i>	<i>(All the labor is used to make chairs.)</i>

Now plot the wood constraint line, using the intercepts  $X_1 = 10$  and  $X_2 = 15$ . Plot the labor constraint line, using the intercepts  $X_1 = 22$  and  $X_2 = 11$ . See Figure 1.

See Figure 1.

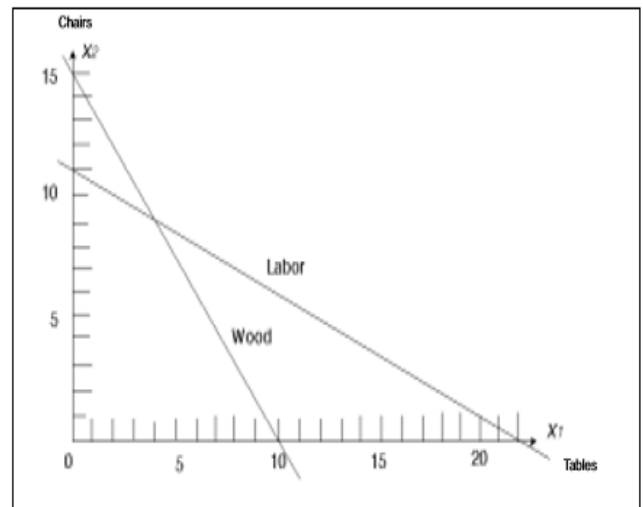


Figure 1.—Wood and labor constraint lines.

**Step 3. Determine the valid side of each constraint line.**

We will use the origin Tables(0,0) to check the valid side for both constraint lines.  $30(0) + 20(0) < 300$  is valid, so we know the side toward the origin (0,0) is the valid side of the wood constraint line.  $5(0) + 10(0) < 110$  also is valid, so we know the side toward the origin (0,0) is the valid side of the labor constraint line. We can draw arrows indicating the valid side of each constraint line. See Figure 2.

We could have chosen any point to test for the valid side of the line. For example, setting  $X_1 = 20$  and  $X_2 = 10$  (clearly on the other side, away from the origin) for the wood constraint line, we get  $30(20) + 20(10) < 300$ , which is not valid. In other words, there simply isn't enough wood to make 20 tables and 10 chairs.

**Step 4. Identify the feasible region.**

The feasible region is the area on the valid side of both constraint lines. Any point located on the invalid side of a constraint line is infeasible. Because of the nonnegativity conditions, the feasible region is restricted to the positive quadrant. See Figure 2.

**Step 5. Plot two objective function lines to determine the direction of improvement.**

First, we'll arbitrarily set profit,  $Z = 48$ , and then set profit,  $Z = 72$ . We'll find the  $x$  and  $y$  intercepts when  $Z = 48$  and when  $Z = 72$ , and plot the two lines.

Set $Z = 48$	Set $Z = 72$
Set $X_2 = 0$ and solve for $X_1$	Set $X_2 = 0$ and solve for $X_1$
$48 = 6(X_1)$	$72 = 6(X_1)$
$48/6 = X_1$	$72/6 = X_1$
$X_1 = 8$	$X_1 = 12$

Next:

Set $X_1 = 0$ and solve for $X_2$	Set $X_1 = 0$ and solve for $X_2$
$48 = 8(X_2)$	$72 = 8(X_2)$
$48/8 = X_2$	$72/8 = X_2$
$X_2 = 6$	$X_2 = 9$

Now plot the objective function lines when  $Z = 48$  and  $Z = 72$ . See Figure 3. We can see from the two objective function lines that as we move away from the origin  $(0,0)$ ,  $Z$  increases.

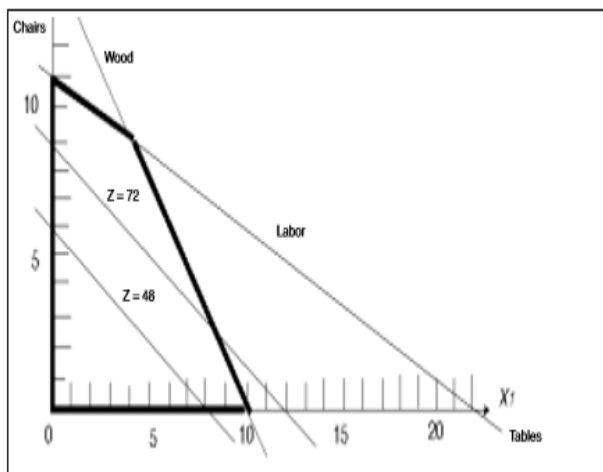


Figure 3.—Determining direction of increasing value.

**Step 6. Find the most attractive corner.**

Since we want to maximize  $Z$ , we will draw a line parallel to the objective function lines that touches the last point in the feasible region [6]

while moving away from the origin. This identifies the most attractive corner, which gives us the amounts of wood and labor that will result in the maximum profit (maximize  $Z$ ). Thus, it represents the optimal solution to the problem (Figure 4).

**Step 7. Determine the optimal solution by algebraically calculating coordinates of the most attractive corner.**

The most attractive corner lies at the intersection of the wood and labor constraint lines. Therefore, coordinates for the most attractive corner can be found by simultaneously solving the constraint equations (wood and labor):

$$30X_1 + 20X_2 = 300 \text{ (wood)}$$

$$5X_1 + 10X_2 = 110 \text{ (labor)}$$

To do so, multiply the labor equation by  $-2$  and add it to the wood equation so the  $X_2$  variable becomes zero and we can solve for  $X_1$ .

$$\begin{array}{r} 30X_1 + 20X_2 = 300 \quad \text{(wood)} \\ -2(5X_1 + 10X_2 = 110) \quad \text{(labor)} \\ \hline 20X_1 + 0 = 80 \\ X_1 = 4 \text{ tables} \end{array}$$

Next, substitute the number of tables calculated above into either of the constraint equations to find the number of chairs. For this example, we will substitute into both equations to illustrate that the same value is found.

<b>Wood constraint</b>	<b>Labor constraint</b>
$30(4) + 20X_2 = 300$	$5(4) + 10X_2 = 110$
$120 + 20X_2 = 300$	$20 + 10X_2 = 110$
$20X_2 = 300 - 120$	$10X_2 = 110 - 20$
$20X_2 = 180$	$10X_2 = 90$
$X_2 = 180/20$	$X_2 = 90/10$
$X_2 = 9 \text{ chairs}$	$X_2 = 9 \text{ chairs}$

Thus, the company's optimal solution is to make four tables and nine chairs. In this case, you could read this solution off the graph (Figure 4) by

finding the values on the x and y axes corresponding to the most attractive corner. However, when the most attractive corner corresponds to an optimal solution with fractions, it is not possible to read directly from the graph. For example, the optimal solution to this problem might have been 3.8 tables and 9.2 chairs, which we probably would not be able to read accurately from the graph.

**Step 8. Determine the value of the objective function for the optimal solution.**

Plug in the number of tables and chairs and solve for Z:

$$Z = \$6(4) + \$8(9) = \$96$$

Thus, we find that maximum profit of \$96 can be obtained by producing four tables and nine chairs.

**4. Limitations to linear programming**

As we saw in an earlier example, there is no guarantee of an integer (whole number) solution. The furniture company can't sell 0.444 of a table. We thought of this as work in progress for this particular problem. An optimal solution for another problem might be to buy 6.75 trucks. Obviously, you can buy either 6 or 7 trucks, but not 0.75 of a truck. In this case, rounding up (and for many non-integer answers, rounding either up or down) offers a practical solution. However, if you're studying the possibility of building a new plant addition, a variable can take on the value of 0 or 1, so a fractional answer is useless. Either you build the addition or you don't. Fortunately, methods called *integer programming* can handle these types of problems.

When using LP, there is no way to deal with uncertainty. We formulate the problem with the assumption that we know all of the values for costs, profits, constraints, etc. In real-life problems, these variables often are unknowns. Methods such as *chance constrained programming* or *linear programming under uncertainty* can be used in these situations.

Another limitation is the assumption of linearity. Often, the objective and constraints are not related linearly to the variables. In some problems, a nonlinear relationship can be forced to be a linear relationship without losing too much integrity in the original problem. For other problems, *nonlinear programming* techniques are available.

Integer programming, linear programming under uncertainty, and nonlinear programming are beyond the scope of these publications. If you wish to learn more about these topics, see the resources listed in the "For more information" section.

**References**

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