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Generalized Differential Transformation Method for Solving Volterra's population growth model of Fractional order

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Abstract

In this paper, the Generalized Differential Transformation Method (GDTM) is used to solve volterra's model for population growth of a species within a closed system. This model is a nonlinear integro-differential equation. Using the prescribed method for this real life application shows it so fast and accurate.. The results demonstrate the applicability and accuracy of the technique.

Key Words: integro-differential equations, ,fractional calculus, generalized differential transformation method, Volterra's population growth model.

Introduction:

There are various applications of Fractional calculus in real life. In practically all areas of science and engineering [28]. Fractional calculus can be defined as a field of applied mathematics that deals with derivatives and integrals of arbitrary orders[15] . In recent years mathematical models including fractional derivatives and integrals have started to be used in different areas ,and it success, provide good rustles in every filed.Many applications in science and engineering are described by integral equations or integro-differential equations[28]. One of these application is Volterra's population growth model (non linear integro-differential equation) which is solved using generalized differential transformation method.The application of differential transform method is successfully extended to obtain analytical approximate solutions to linear and nonlinear integro- differential equations of fractional order. The is Volterra's population growth of a species within a closed system was solved using many numerical methods such as A domain decomposition method used by Wazwaz to get analytic approximations and Pad´e approximants for Volterra's population model[1]. He presented an analytical approximation solution of Volterra's model population growth of a same species in closed system. Also the Volterra's population growth of a species in closed system was solved in [30]using variational iteration method(VIM). Differential transform (DT) has taken the shape of an important and convenient tool. Roughly speaking two decades have elapsed since the dawn of applications of differential transform method in numerical methods for solving differential equations. It tries to formalize the Taylor series, definitely with a different approach[24]. In(1980) G.E. Pukhov used differential transform in numerical methods to solve fractional differential equations for the first time[10],[11]. By utilizing this method the given differential equation and related initial conditions are changed into a recurrence equation[12],[13],[24].In (1986) Zhou used

(DTM) in electric circuit analysis,[31]. Since then, (DTM) was success-fully applied for a large variety of problems. The method has been applied to solved different problems with approximations converging rapidly to exact solutions for along 15years later many researchers employed DTM to solve all types of equations[2].[7], [8],[17],[18],[22].

This paper is organized as: Section one shows some basic concept that we need. section tow described GDTM Technique. Illustrated example solved in tow cases with table of results and diagrams in section three . conclusions are proposed in section four.

1. Basic Concept

1.1 The Fractional Derivatives

To describe the Fractional derivatives we give the basic definitions of fractional derivative:

Definition (1.1)(Caputo Fractional Derivatives D_C^{α}),[5],[16],[27]:

Let $f(t) \in C_{\mu}^{n}$ [25] that is defined on the closed interval [a,b], the Caputo Fractional derivative of order $\alpha > 0$ of f is defined by:

$$D_c^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau , & n-1 < \alpha < n , \quad n \in N \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \quad n \in N \end{cases} \quad (1.1)$$

Definition (1.2)(Riemann-Liouville Fractional Intrgrals),[23],[17]:

Let $f(t) \in C_{\mu}^{n}$ that is defined on the closed interval [a,b], Riemann-Liouville Fractional integral of order$\alpha > 0$ of f is defined by:

$$J^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau \quad (1.2)$$

Definition(1.3)(Gamma function),[20], [21],[27]:

The complete gamma function $\Gamma(t)$ is also known as generalized Factorial function. It is defined by using the following integral:

$$\Gamma(t) \ = \int_0^{\infty} S^{t-1} e^{-S} dS , \quad t > 0 \quad , S \text{ any variable} \quad (1.3)$$

(1.4)(Properties of Gamma function), [21],[27]:

(1) $\Gamma(t+1) = t\Gamma(t) \quad t > 0$

(2) $\Gamma(t) = (t-1)!$t is positive integer, convention: $0! = 1$

(3)$\Gamma\frac{1}{2} = \sqrt{\pi}$

Definition(1.5)(Differential transform DT),[2],[3],[10],[26];

The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method obtains a polynomial series solution by means of an iterative procedure.

Fractional Differential transform can be defined as:

$$F(k)=\begin{cases} \frac{k}{\alpha}\in Z^+, & \frac{1}{\left(\frac{K}{\alpha}\right)!}\left[\frac{d^{\frac{k}{\alpha}}f(x)}{dx^{k/\alpha}}\right]_{x=x_0} \\ \frac{k}{\alpha}\notin Z^+, & 0 \end{cases} \text{for} \quad k=0,1,\ldots,(q\alpha-1) \qquad (1.4)$$

Where α is the order of fractional derivative[7],[9]: .

And We define the generalized differential transform of the kth derivative of function f(t) in one variable as [14]: $F(k)=\frac{1}{\Gamma(\alpha k+1)}[\left(D_{t_\circ}^{\alpha}\right)^k f(t)]_{t=t_\circ}$ (1.5)

where $(D_{t_\circ}^{\alpha})^K = D_{t_\circ}^{\alpha}.D_{t_\circ}^{\alpha}\ldots\ldots D_{t_\circ}^{\alpha}$,k-times and the differential inverse transform of $F(k)$is defined as: $f(t)=\sum_{k=0}^{\infty}F_{\alpha}(k)(t-t_\circ)^{\alpha k}$ (1.6)

(1.6)(Some properties of GDTM),[7],[8],[9],[29]:

1-If $f(t)=g(t)\pm h(t)$, then $F(k)=G(k)\pm H(k)$.

2-If f(t) = ag(t), then $F(k)=aG(k)$, where a is a constant.

3- If f(t) = g(t) h(t), then $F(k)=\sum_{l=0}^{k}G(l)\ H(k-l)$

4- If f(t)= $D_{t_\circ}^{\alpha}g(t)$, then $F(k)=\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)}G(k+1)$

5-If f(t)=$(t-t_\circ)^{\beta}$, then $F(k)=\delta\left(k-\frac{\beta}{\alpha}\right)$,where $\delta(k)=\begin{cases}1 \text{ if } k=0\\ 0 \text{ if } k\neq 0\end{cases}$

6-If f(t)=$\int_{t_\circ}^{t}g(t)dt$, then $F(k)=\frac{G\left(k-\frac{1}{\alpha}\right)}{\alpha k}$where $k\geq\frac{1}{\alpha}$

7-If f(t)=g(t)$\int_{t_\circ}^{t}h(t)dt$ then$F(k)=\sum_{k_1=\frac{1}{\alpha}}^{k}\frac{H\left(k-\frac{1}{\alpha}\right)}{\alpha k_1}G(k-k_1)$where $k\geq\frac{1}{\alpha}$

1.2 Volterra's population model[1],[4],[19]

The Volterra's population model for population growth of a species within a closed system is characterized by nonlinear Volterra integro differential equation.

$$\frac{dP}{dt}=aP-bP^2-cP\int_0^t P(x)dx, \quad P(0)=P_0 \qquad (1.7)$$

Where P=P(T) denotes the population at time T,

a,b and c are constants and positive parameters

a> 0 is birth rate coefficient ,b> 0 is the crowding coefficient, c> 0 is the toxicity coefficient, and p_0 is the initial population.

Many time scales and population scales may be applied. However we apply the scale time and population suggesting the non-dimensional variables, $T=\frac{cT}{b}$, $u=\frac{bp}{a}$,to obtain the non dimensionalVolterra's population growth model

$$k\frac{du}{dt}=u-u^2-u\int_0^t u(x)dx \qquad u(0)=u_0 \qquad (1.8)$$

where u =u(t) is the scale population of identical individuals at a time t, and the non-dimension parameter $k=c/(ab)$ is a prescribed parameter.

2. Solving non Linear Volterra Integro-Differential Equations of Fractional order(NL-FVIDE) using GDTM Technique,[6],[7],[8]:

The technique that we used is the differential transform method (DTM),which is based on Taylor series expansion. It is introduced by Zhou[60] in a study about electrical circuits.

Consider the NL-FVIDE. $D^{\beta}u(t)=f(t)+\lambda\int_0^t K(t,x,u)u^i(x)dx \qquad (1.9)$

with initial condition u(0)= a , $0<\beta\leq 1$,$\lambda\in\mathbb{R}$, i=1,…,n ,$n\in\mathbb{Z}^+$

$D^{\beta}u(t)$denotes the Caputo fractional derivative of order β for u(t). f(t) is continuous function.

To solve the equation (1.9) using GDTM, one can take the differential transform for both sides of equation (1.9). According to GDTM's properties in (1.6),the terms of equation (1.9) can be transform as following:

1- $D^{\beta}u(t)$ transformed to $\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)}U(k+\frac{\beta}{\alpha})$

2- f(t)transformed toF(k)

3-$\lambda\int_0^t K(t,x,u)u^i(x)dx$ transformed to

$$\frac{1}{\alpha k}\sum_{k_{i-1}=0}^{k-\frac{1}{\alpha}}\sum_{k_{i-2}}^{k_{i-1}}\dots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_1(k_1)U_2(k_2-k_1)..U_{i-1}(k_{i-1}-k_i-\frac{1}{\alpha})*\lambda*F\{K(t,x,u)\}\forall\, i=1,\dots,n$$

In this part of the transform, k satisfies that $k\geq\frac{1}{\alpha}$,taking into consideration what is suitable for each function in terms of transformation.

Next we can characterize the new equation to find $U(k+\frac{\beta}{\alpha})$, $k=0,\dots,n$.

such that

$$U(k+\frac{\beta}{\alpha})=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}[F(k)+\frac{1}{\alpha k}\sum_{k_{i-1}=0}^{k-\frac{1}{\alpha}}\sum_{k_{i-2}}^{k_{i-1}}\dots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_1(k_1)U_2(k_2-k_1)..U_{i-1}(k_{i-1}-k_i-\frac{1}{\alpha})]*\lambda*F\{K(t,x,u)] \qquad (1.10)$$

Now we have two cases:

First case When $\beta=\alpha=1$

To transform the initial condition of (1.9) we need to use the following relation at t=a

$$U(k_\circ)=\begin{cases}U(k_\circ)=\frac{1}{\alpha k_\circ}*\frac{du}{dt} & \text{If } \alpha k\in\mathbb{Z}^+\\ U(k_\circ)=0 & \text{If } \alpha k\notin\mathbb{Z}^+ \quad \forall\, k_\circ=0,\dots,n\end{cases}$$

where $k_\circ=\frac{\beta}{\alpha}-1$,at t=0 .

It is clear that $k_\circ=0$ in this case, and by substituting $\beta=\alpha=1$ the value of $U(k+\frac{\beta}{\alpha})$ will be U(k+1) . Next substituting k values in the obtained equation $\forall\, k=0,\dots,n$. One can find the values of U(k_i+1) $\forall\, i=0,\dots,n$ which present the transformed series of U(k_i+1) ,after this depending on the derivations of equation (1.6) in properties (1.6).Taking the inverse transform by using the following relation

$$u(t)=\sum_{k=0}^{\infty}U(k)(t-t_\circ)^{\alpha k}\; t_\circ=0 \quad \alpha=1$$

$$u(t) = \sum_{k=0}^{\infty} U(k)t^{\alpha k}$$

We get the semi analytic solution for equation (1.9) in series form.

Second case When β is fractional

In this case selecting α must satisfie:

- $\alpha \ \leq \beta - 1$.
- $\frac{\beta}{\alpha} \ \in \mathbb{Z}^{+}$

By the same way we can substitute values of β and α in U(k+ $\frac{\beta}{\alpha}$), $k = 0, \dots, n$. and apply the same steps to obtain the transformed initial condition . Then, we take k values ∀ $k = 0, \dots, n$, to find U(k_i+ $\frac{\beta}{\alpha}$)∀ $i = 0, \dots, n$. after this, we take the inverse transform

$$u(t) = \sum_{k=0}^{\infty} U(\,k)(t - t_{\circ})^{\alpha k} \qquad t_{\circ} = 0 \quad , \alpha \text{ is fractional}$$

$$u(t) = \sum_{k=0}^{\infty} U(k)t^{\alpha k}$$

to obtain the approximation solution for the original equation (1.9) in series form. Next to illustrate the solution procedure and show the feasibility and efficiency of the GDTM we have applied the method . Next, we solve Volterra's population equation using GDTM in two different cases.

3. The illuminative application example,[1]

Consider the following Volterra's population initial value equation

$$\frac{du}{dt} = 10u(t) - 10\,u^2(t) - 10u(t)\int_0^t u(x)dx \qquad (1.11)$$

With initial condition u(0) = 0.1

Assuming that the order of it divertive is β such that 0< $\beta \leq 1$ the general formwill be:-

$$D^B u(t) = 10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx \qquad (1.12)$$

The obtained equation (1.12) is called Volterra's population initial value equation of Fractional order .

By applying the technique described in section 2 one can transform each part of equation (1.12)using the properties of GDTM in (1.6). The equation will be:

$$U\left(k + \frac{\beta}{\alpha}\right) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}\left[10U(k) - 10\sum_{k_1=0}^{k} U(k_1)U(\,k - k_1) - 10\sum_{k_1=1/\alpha}^{k} \frac{1}{\alpha k_1}\,U\left(\,k_1 - \frac{1}{\alpha}\right)U(k - k_1)\right] \ (1.13)$$

First case:

To obtain the first solution, put β = α= 1

which means $k_{\circ} = 0$ then U(0) = 0.1

Substituting β and α in equation (1.13) we get

$$U(k+1) = \frac{r(k+1)}{r(k+2)}\left[10U(k) - 10\sum_{k_1=0}^{k} U(k_1)U(k-k_1) - 10\sum_{k_1=1}^{k}\frac{1}{k_1}\,U(\,k_1-1)U(k-k_1)\right] (1.14)$$

Substituting the values k in equation (1.14) ∀ $k = 0,1,2,\dots$

For k = 0 then U(1) = 0.9 , For k = 1 then U(2) = 3.55

and by the same way we can find U(3),U(4) ,….. Now applying the inverse transform to get first solution formed in series form :-

$$u(t)=\sum_{k=1}^{\infty} U(k)(t - t_0)^{\propto k} \ t_0=0, \propto= 1\ , \quad u(t) = \sum_{k=1}^{\infty} U(k)t^{k}$$

$$u(t) = U(0)t^0 + U(1)t + U(2)\,t^2 + U(3)t^3 + U(4)t^4 + U(5)\,t^5 + \,.....+ U(n)\,t^n + ...$$
$$u(t) = 0.1 + 0.9t + 3.55t^2 + 6.466666667t^3 + (-5.4750000)t^4 + (-68.6000000)t^5 + ...$$

Second case:

For this case one can select the value of β as:

$\beta = 0.5 = ½$ and $\alpha = 0.5 = ½$. , Which means $k_\circ = 0$ then $U(0) = 0.1$

By substituting β and α values in equation (1.13) we get

$$U(k+1) = \frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}+1\right)}\left[10U(k) - 10\sum_{k_1=0}^{k} U(k_1)U(k-k_1) - 10\sum_{k_1=5}^{k}\frac{5}{k_1}U(k_1-5)U(k-k_1)\right] \quad (1.15)$$

Once again ,substituting k values in equation $(1.15)\,\forall\, k = 0,1,2,.....$

For k = 0 then U(1) = 1.0155413 ,For k = 1 then U(2) = 7.200000003.

And by the same way we can find U(3) ,U(4),….. Again applying the inverse transform of equation (1.15) to get approximation solution for equation(1.13) formed in series form:

$$u(t)=\sum_{k=1}^{\infty} U(k)(t-t_0)^{\propto k} \quad t_0=0\,, \propto=\tfrac{1}{2}\,, u(t) = \sum_{k=0}^{\infty} U(k)t^{\frac{1}{2}k}$$
$$u(t) = U(0)t^0 + U(1)t^{\frac{1}{2}} + U(2)\,t + U(3)\,t^{\frac{3}{2}} + U(4)\,t^2 + U(5)\,t^{\frac{5}{2}} + \,....U(n)\,t^{\frac{n}{2}} + \cdots$$
$$u(t) = 0.1 + (1.0155413)t^{\frac{1}{2}} + (7.200000003)\,t + (35.4963711)\,t^{\frac{3}{2}} + (90.6470388)\,t^2 + (-327.3434819)\,t^{\frac{5}{2}} + \,..........U(n)\,t^{\frac{n}{2}} + ...$$

Finding an arbitrary value of t for any fractional divertive order β and calculate it:

For example $\beta = 0.8$ then $\alpha = 0.2$. By substituting β and α values in equation (1.13) we get

$$U(k+4) = \frac{\Gamma\left(\frac{k}{5}+1\right)}{\Gamma\left(\frac{k}{5}+\frac{4}{5}+1\right)}\left[10U(K) - 10\sum_{k_1=0}^{k} U(k_1)U(k-k_1) - 10\sum_{k_1=5}^{k}\frac{1}{\alpha k_1}U(k_1-5)U(k-k_1)\right] \quad (1.16)$$

For k=0 then U(4) = 0.9663041 , and by the same way we can find U(5),U(6),…,U(10) Applying the inverse transform to get approximation solution formed in series form:

$$u(t)=\sum_{k=1}^{\infty} U(k)(t-t_0)^{\propto k} \quad t_0=0, \propto=\tfrac{1}{5}\;,\quad u(t) = \sum_{k=0}^{\infty} U(k)t^{\frac{1}{5}k}$$

$$u(t) = U(0)\,t^0 + U(1)t^{\frac{1}{5}} + U(2)t^{\frac{2}{5}} + U(3)\,t^{\frac{3}{5}} + U(4)\,t^{\frac{4}{5}} + U(5)\,t + \,....+ U(n)\,t^{\frac{n}{5}} + \cdots$$

For t=0.6 , $\beta = 0.8$,$\alpha = 0.2$ we can obtain u(t)=2.96623828

This value of u(t) is illustrated colored as red in the table below, we can find the other entries values of the table by the same way.

Table shows t Values of the numerical solution for application example solved using GDTM considering different values for the fractional divertive $\boldsymbol{\beta}$

Values of t	$\propto = \boldsymbol{\beta} = \mathbf{1}$		$\boldsymbol{\beta} = \mathbf{0.5}$	$\boldsymbol{\beta} = \mathbf{0.75}$	$\boldsymbol{\beta} = \mathbf{0.8}$	$\boldsymbol{\beta} = \mathbf{0.9}$
	N=3	N=10	N=10			
0.0	0.10000000	0.10000000	0.10000000	0.10000000	0.10000000	0.10000000
0.1	0.23260000	0.23119930	53.78617051	0.54235712	0.37965470	0.21780756
0.2	0.47680000	0.43940877	2017.08217184	1.33664444	0.75014162	0.31983668
0.3	0.87220000	0.69588502	15822.31041389	2.52768864	1.20249325	0.41665204
0.4	1.45840000	1.98774368	67351.53595082	4.13755895	1.72679680	0.51022975
0.5	2.27500000	12.48730125	206259.20514487	6.18422998	2.31635122	0.60147140
0.6	3.36160000	69.39833506	513747.86782346	8.68319025	2.96623828	0.69089361
0.7	4.75780000	298.36324364	1110316.14080792	11.64816416	3.67264025	0.77883058
0.8	6.50320000	1049.82484214	2163487.57978413	15.09153403	4.43247540	0.86551579
0.9	8.637400000	3176.78829890	3895525.34331468	19.02461687	5.24318372	0.95112123
1.0	11.20000000	8552.38014334	6591135.73736971	23.45785881	6.10259144	1.03577872

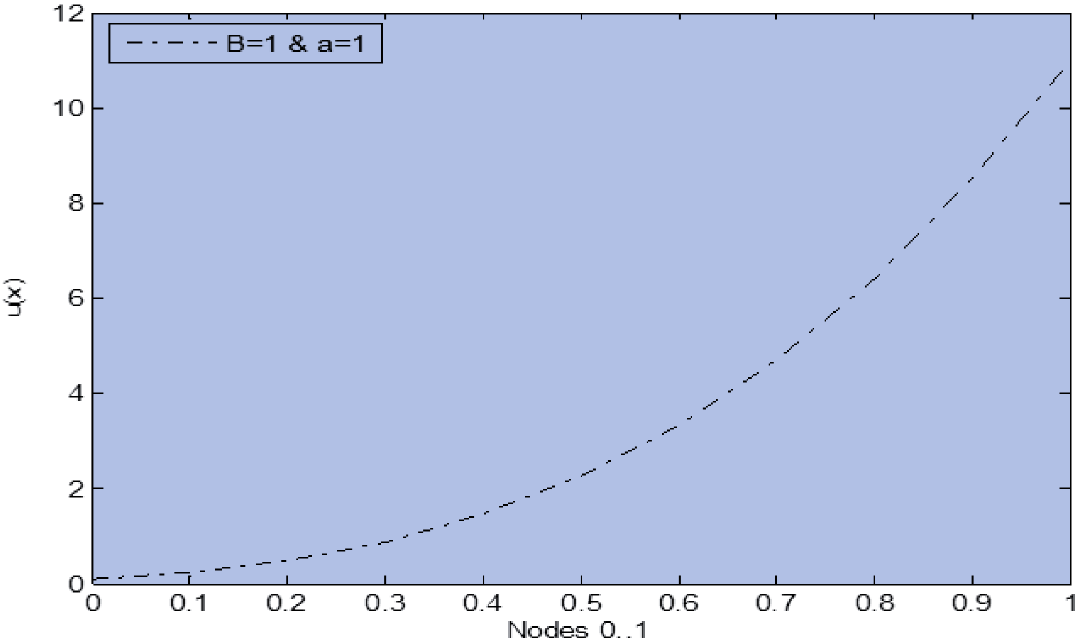

Figure (1): Approximation solution for application example using GDTM when N=3 , β=1

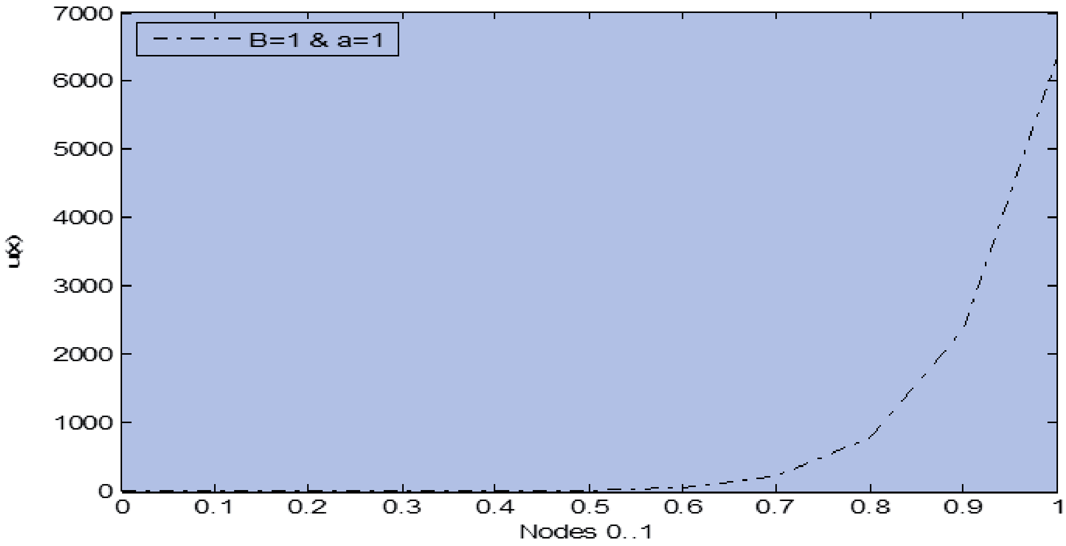

Figure (2): Approximation solution for application example using GDTM when N=10 , β=1

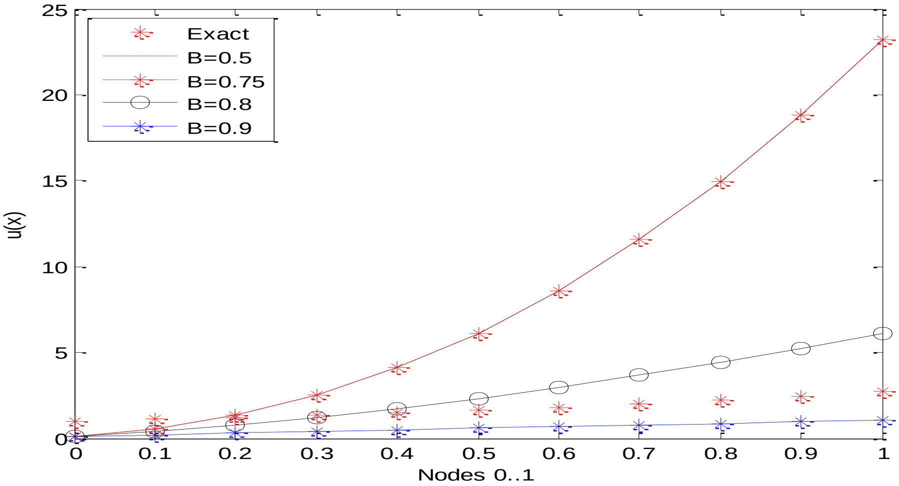

Figure (3): Comparison between the approximate solutions for application example using GDTM when N=10 for different values for the fractional order divertive β

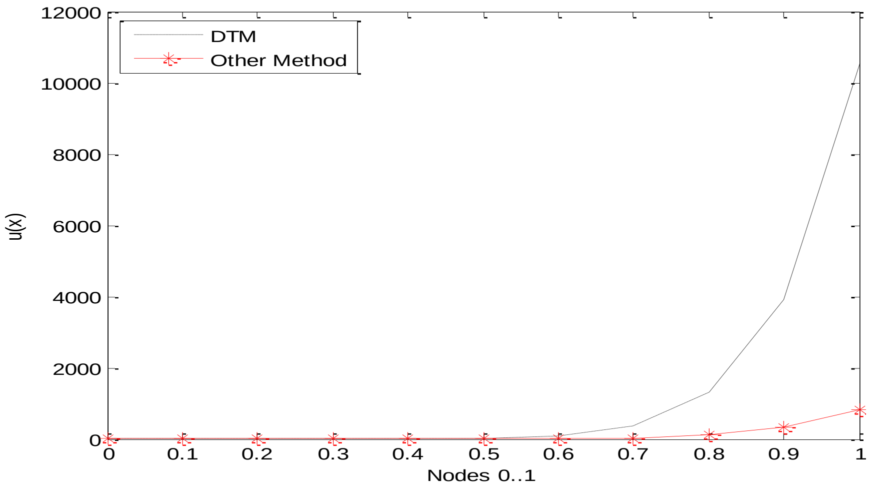

Figure (4): Comparison between the approximation solution for application example using GDTM and other methods when N=10 , β=1

4. Conclusion:-

Generalized Differential Transformation Method an efficient method to solve many applications in science and engineering . This method has been successfully applied to find the approximate solution of the fractional non linear Volterra integro- differential equations .The fractional order models for many real life applications are more accurate than integer- order models. The GDT method provides more realistic series solutions that converged very rapidly in real physical problem.

References

[1]. Abdul-Majid Wazwaz,"linear and non linear integral equations methods and applications", higher education press,Beijing and springer-verlag berlin Heidelberg,2011.

[2]. Arikoglu A., Ozkol I.," Solutions of integral and integro-differential equation systems by using differential transform method", Computers and Mathematics with Applications 56,pp. 2411–2417, 2008.

[3].Arikoglu A. and Ozkol I., " Solution of fractional integro-differential equations by using

fractional differential transform method" , Chaos, Solution and Fractals 40 pp. 521-529,2009.

[4].B. M. Pandya, D. C. Joshi," Solution of a Volterra's Population Model in a Bernstein Polynomial Basis", Applied Mathematical Sciences, Vol. 5, No. 69, pp.3403 – 3410, 2011.

[5]. Diethelm K."The Analysis of Fractional Differential Equations and Application-Oriented Exposition Using Differential Operators of Caputo Type",Springer-Verlag Berlin Heidelberg, 2010.

[6].El-Shahed M.," Application of differential transform method to non-linear oscillatory systems", Communications in Nonlinear Science and Numerical Simulation, No. 13,PP. 1714–1720,2008.

[7].Ertürk V. S., and Momani S, "Solving system of fractional differential equations using differential transform method" , Journal of Computational and Applied Mathematics 215, pp. 142-151,2008.

[8]. Ertürk V. S., and Momani S, " On the Generalized Differential Transform Method Application to Fractional Integro-Differential Equations", IDOSI Publications, 2010.

[9]. Erturk V.S., Momani S., Odibat Z.,"Application of generalized differential transform method to multi-order fractional differential equations", Communications in Nonlinear Science and Numerical Simulation ,No.13,pp. 1642–1654,2008.

[10].G. E. Pukhov, "Differential transforms of functions and equations", in Russian,Naukova Dumka, Kiev, 1980.

[11]. G. E. Pukhov," Expansion formulas for differential transforms", Cybern. Syst.Anal. , no.17,1981 .

[12].G. E. Pukhov, "Differential transforms and circuit theory", Int. J. Circ. Theor.App. , No.10,1982.

[13]. G. E. Pukhov," Differential transformations and mathematical modeling ofphysical processes", in Russian, Naukova Dumka, Kiev, 1986.

[14]. Gorecki A., "Application of the Fractional Calculus to the Model of Populations Growth", Math. Comput. Modeling, Vol. 8, PP. 15-25, 1989.

[15].Hilfer R., "Fractional Calculus in Physics", First Edition, World Scientific Publishing, Singapore, New Jersey, London and Hong Kong, 2000.

[16].H. Vic Dannon," the Fundamental Theorem of the Fractional Calculus and Meaning of Fractional Derivatives", Gauge Institute Journal, Volume 5, No1, February 2009.

[17].Jang M.J., Chen Ch.," On solving the initial-value problems using the differential transformation method", Applied Mathematics and Computation, No.115 ,pp.145-160, 2000.

[18].Kacprzyk J., "Fractional Linear Systems and Electrical Circuits", Polish Academy of Sciences, Poland,Springer International Publishing Switzerland, Vol.13,2015.

[19]. Khodabin M., Maleknejad K., Asgari M.,"Numerical solution of a stochastic population growth model in a closed system", Khodabin et al.Advances in Difference Equations, no.130, 2013.

[20]. Kilbas A.A., Srivastava H.M. and Trujillo J.J., "Theory and Applications of Fractional Differential Equations", Elsevier, Amsterdam, 2006.

[21]. Leon M. Hall," Special Functions", University of Missouri-Rolla,1995.

[22]. Liu H., Song Y.," Differential transform method applied to high index differential–algebraic equations", Applied Mathematics and Computation ,No.184,pp.748–753, 2007.

[23]. Loverro A, "Fractional calculus: History, Definitions and pplications for the Engineer" Areport, University of Notre Dame May 8,2004.

[24]. M.M. Rashidi, S.A. Mohimanian , N. Laraqi, "A semi-analytical solution of micro polar flow in a porouschannel with mass injection by using differential transform method", Nonlinear Analysis: Modelling and Control, Vol. 15, No. 3,pp. 341–350,2010.

[25]. Nishimoto K, " Fractional Calculus: " Integrations and Differentiations of Arbitrary Order" , Descartes Press Co. Koriyama Japan, 1983.

[26]. Odibat Z. M., " Differential transform method for solving Volterra integral equation with separable kernels" , Mathematics and Computer Modeling 48, pp.1144-1149,2008 .

[27]. Oldham K. Band Spanier J, "The fractional calculus: theory and order"Academic applications of Differentiation and integration to arbitrary press, New york and London, 1974.

[28]. Ross B. "fractional calculus and its applications" Lecture Notes in Mathematics, proceeding of the international conference Held at the University of new Haven, Vol 457,1994.

[29]. Salah H. Behiry, Saied I. Mohamed,"Solving high-order nonlinear Volterra-Fredholm integro-differential equations by differential transform method ", Natural Science ,Vol.4, No.8,pp. 581-587, 2012.

[30].Yaseen M.S. Al-Wesabi, Abdullah A. Dahawi , Yahaya Daniel, Ali H.M. Murid," Analytical solution of Volterra's population model using Variation Iteration Method", International Conference of Recent Trends in Information and Communication Technologies IRICT,pp. 413-423,2014.

[31]. Zhou J. K., " Differential Transformation and its Applications for Electrical Circuits" , Huazhong University Press,Wuhan, China, 1986.