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Solving linear Volterra–Fredholm integro- differential equations of fractional order by using Generalized Differential Transform Method

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ABSTRACT

In this paper , Generalized differential transform method (GDTM) is applied to solve linear integro-differential volterra –fredholm equation of fractional order .Which has an exact solution when the order of the divertive is equal to 1. The fractional derivative is considered in the Caputo sense. Given example has been solved in two different cases. The results make known the applicability and accuracy of the practice.

Key Words: integro-differential equations, , fractional calculus, Generalized differential transformation method, linear Integro-Differential Volterra –Fredholm equation .

Introduction:

In recent years, there has been incessantly reformed attention in integro -differential equations .Many mathematical models of physical phenomena create integro-differential equations e.g fluid dynamics, biological models and chemical kinetics [2],[5]. Electromagnetism, acoustics,viscoelasticity, electrochemistry and material science are also well-described by fractional integro-differential equations [1],[4],[5],[7]. Due to the frequent applications of fractional calculus (a mathematical branch investigating the properties of derivatives and integrals of non-integer orders) [8] , in varied fields, the solution techniques for fractional differential equations of a range of forms and classes, carry on to draw increasing attention from many researchers. Since the fractional calculus comparatively a new subject in mathematics, Many numerical methods and techniques give an acceptable approximation solution or semi- analytic[21],[22]. Some of these methods are Adomian Decomposition method (ADM), Variational Iteration method (VIM), Homotopy analysis method(HAM), homotopy perturbation method (HPM) and Differential Transform method (DTM), [18], [19].

Differential Transform (DT) has taken the shape of an important and convenient tool.In(1980) G.E. Pukhov used differential transform in numerical methods to solve fractional differential equations for the first time[12],[13]. In (1986) Zhou used (DTM) in electric circuit analysis[17],[28]. Since then, (DTM) was success-fully applied for a large variety of problems. A few researchers involved with fractional Volterra-Fredholm integro-differential equations and treated this type numerically and this study present for the first time the Generalized differential transformation method as a numerical tool to solve this type of equations so we structure a technique that disclaimed later and give a good example to show the obtained results with table of different values of t in different cases of fractional derivatives of β.

One can characterized the out lines of this paper as: Frist section shows some basic concept that we will need. section tow described GDTM Technique. illustrated example solved in tow cases with table of results and diagrams in section three . conclusions are proposed in section four.

1. basic concept

Definition (1.1)(Caputo Fractional Derivatives D_C^{α}),[8],[15],[25]:

Let $f(t) \in C_{\mu}^{n}$ [23] that is defined on the closed interval [a,b], the Caputo fractional derivative of order α> 0 of f is defined by:

$$D_c^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau \,, & n-1 < \alpha < n\,, \quad n \in N \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n, \quad n \in N \end{cases} \quad (1.1)$$

Definition (1.2)(Riemann-Liouville Fractional Intrgrals),[20],[23],[27]:

Let $f(t) \in C_{\mu}^{n}$ that is defined on the closed interval [a,b], Riemann-Liouville Fractional integral of orderα> 0 of f is defined by:

$$J^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau \quad (1.2)$$

Definition(1.3)(Gamma function),[13], [14],[16],[27]:

The complete gamma function $\Gamma(t)$ is also known as generalized Factorial function. It is defined by using the following integral:

$$\Gamma(t) = \int_0^{\infty} S^{t-1} e^{-S}\, dS\,, \quad t > 0 \quad , S \text{ any variable} \quad (1.3)$$

(1.4)(Properties of Gamma function), [14],[16]:

(1) $\Gamma(t+1) = t\Gamma(t) \quad t > 0$

(2) $\Gamma(t) = (t-1)!$t is positive integer, convention: 0! = 1

(3)$\Gamma\frac{1}{2} = \sqrt{\pi}$

Definition(1.5)(Differential transform DT),[2],[3],[12],[24];

The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method obtains a polynomial series solution by means of an iterative procedure.

Fractional Differential transform can be defined as:

$$F(k)=\begin{cases} \frac{k}{\alpha}\in Z^+, \frac{1}{\left(\frac{K}{\alpha}\right)!}\left[\frac{d^{\frac{k}{\alpha}}f(x)}{dx^{k/\alpha}}\right]_{x=x_0} \text{for } k=0,1,\ldots,(q\alpha-1) \\ \frac{k}{\alpha}\notin Z^+, 0 \end{cases} \quad (1.4)$$

Where α is the order of fractional derivative[4],[6]: .

And We define the generalized differential transform of the kth derivative of function f(t) in one variable as follows,[6]:: $F(k)=\frac{1}{\Gamma(\alpha k+1)}[\left(D_{t_\circ}^{\alpha}\right)^k f(t)]_{t=t_\circ}$ (1.5)

where $(D_{t_\circ}^{\alpha})^K = D_{t_\circ}^{\alpha}.D_{t_\circ}^{\alpha}\ldots\ldots D_{t_\circ}^{\alpha}$,k-times and the differential inverse transform of $F(k)$is defined as follows: $f(t)=\sum_{k=0}^{\infty}F_{\alpha}(k)\left(t-t_\circ\right)^{\alpha k}$ (1.6)

(1.6)(Properties of GDTM),[9],[10],[11],[26]

1-If $f(t)=g(t)\pm h(t)$, then $F(k)=G(k)\pm H(k)$.

2-If f(t) = ag(t), then $F(k)=aG(k)$, where a is a constant.

3- If f(t) = g(t) h(t), then $F(k)=\sum_{l=0}^{k}G(l)\,H(k-l)$

4-If f(t)=$g_1(t)g_2(t),\ldots\ldots..,g_{n-1}(t)g_n(t)$, then

f(x)=$\sum_{k_{n-1=0}}^{k}\sum_{k_{n-1=0}}^{k_{n-1}}\ldots\ldots\sum_{k_{n-1=0}}^{k}\sum_{k_{n-1=0}}^{k}G_1(k_1)G_n(k_2-k_1)\ldots\ldots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1})$

5- If f(t)= $D_{t_\circ}^{\alpha}g(t)$, then $F(k)=\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)}G(k+1)$

6-If f(t)=$\left(t-t_\circ\right)^{\beta}$, then $F(k)=\delta\left(k-\frac{\beta}{\alpha}\right)$,where $\delta(k)=\begin{cases}1 \text{ if } k=0\\0 \text{ if } k\neq 0\end{cases}$

7-If f(t)=$\int_{t_\circ}^{t}g(t)dt$, then $F(k)=\frac{G\left(k-\frac{1}{\alpha}\right)}{\alpha k}$where $k\geq\frac{1}{\alpha}$

8-If f(t)=g(t)$\int_{t_\circ}^{t}h(t)dt$ then$F(k)=\sum_{k_1=\frac{1}{\alpha}}^{k}\frac{H\left(k-\frac{1}{\alpha}\right)}{\alpha k_1}G(k-k_1)$ where $k\geq\frac{1}{\alpha}$

9- If f(t)=$\int_{t_\circ}^{t}h_1(t)h_2(t)\ldots\ldots\ldots h_{n-1}(t)h_n(t)dt$, then

$F(k) = \frac{1}{\alpha k}\sum_{k_{n-1}=0}^{k-\frac{1}{\alpha}} \sum_{k_{n-2}}^{k_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} H_1(k_1)H_2(k_2-k_1)..H_{n-1}\left(k_{n-1}-k_n-\frac{1}{\alpha}\right], k \geq \frac{1}{\alpha}.$

10- If $f(t)=[g_1(t)g_2(t) \dots \dots \dots \dots g_{m-1}(t)g_m(t)] \int_{t_\circ}^{t} h_1(t)h_2(t) \dots . h_{n-1}(t)h_n(t)dt$,

then $F(k) = \sum_{k_i}^{k} \frac{1}{\alpha\, k_1} \sum_{j_{n-1}=0}^{k_1-\frac{1}{\alpha}} \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \sum_{j_1=0}^{k-k_1} \sum_{i_{m-2}=0}^{i_{m-1}} G_1(i_1)\, G_2(i_2-i_1) \dots G_{m-1}(i_{m-1} -$

$i_{m-2})G_m(k-i_{m-1}-k_1) \times H_1(j_1)H_2(j_2-j_1) \dots . . H_{n-1}(j_{n-1}-j_{n-2})H_n(k_1-j_{n-1}-\frac{1}{\alpha})$,

where $k \geq 1/\alpha$.

11- If $f(x) = \int_a^b h\ (t)\ dt$ then $F(k) = \frac{1}{\alpha k} * \ \delta(k) * \frac{G_\alpha\left(k-\frac{1}{\alpha}\right)}{\alpha k}$ where $k \geq \frac{1}{\alpha}$

12- For special functions that may used : If $f(t) = e^{\lambda t}$ then $F(k)=\frac{\lambda^k}{k!}$

(2) Solving Linear Volterra-Fredholm Integro-Differential Equations of Fractional order Using GDTM Technique

This section gives GDTM technique to solve linear Volterra-Fredholm integro- differential equations of fractional order. Consider the L-FFVIDE .

$$D_c^{\beta} u(t) = f(t) + \lambda_1 \int_0^t K_1(t,x)u\ (x)dx + \lambda_2 \int_0^1 K_2(t,x)u\ (x)dx \qquad (2.1)$$

With initial condition $u(0)= a$, $0< \beta \leq 1$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $D_c^{\beta} u(t)$ denotes the Caputo fractional derivative of order β for $u(t)$, $f(t)$ is contiuous function with $f(t) \in C_\mu^n, t \in [a,b]$. To solve equation (2.1) using GDTM ,one can take the differential transform for both sides of equation (2.1).According to GDTM' s properties in (1.6),the terms of equation (2.1) can be transformed as follows:

1- $D_c^{\beta} u(t)$ transformed to $\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} U(k + \frac{\beta}{\alpha})$

2-$f(t)$transformed to $F(k)$

3-$\lambda_1 \int_0^t K_1(t,x)u\ (x)dx$ transformed to $\frac{1}{\alpha k} U\left(k_1 - \frac{1}{\alpha}\right) \lambda_1 F\{K_1(t,x)\}$,as additional step to transform Fredholm equation part is as follows:

4- $\lambda_2 \int_0^1 K_2(t,x)u\ (x)dx$ transformed to $\delta(k)\frac{1}{\alpha k} U\left(k_2 - \frac{1}{\alpha}\right) \lambda_2 F\{K_2(t,x)\}$

Once again, these two parts of the transform k satisfy $k \geq \frac{1}{\alpha}$, keeping in mind what is suitable for each function in terms of transformation. Next ,we can characterize the new equation to find $U(k+\frac{\beta}{\alpha})$, $k = 0, \dots, n$. such that

$$U(k+\frac{\beta}{\alpha}) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)} [F(k) + \frac{1}{\alpha k} U\left(k - \frac{1}{\alpha}\right) \lambda_1 F\{K_1(t,x)\} + \delta(k) \frac{1}{\alpha k} U\left(k_2 - \frac{1}{\alpha}\right) \lambda_2 F\{K_2(t,x)\}]$$

(2.2)

For the cases solution for the types of the derivative order , we have two cases:

First case When $\beta = \alpha = 1$

To transform the initial condition of equation (2.1) we need to use the following relation at t=a

$$U(k_\circ) = \begin{cases} \text{If } \alpha k \in \mathbb{Z}^+ & U(k_\circ) = \frac{1}{\alpha k_\circ} * \frac{du}{dt} \\ \text{If } \alpha k \notin \mathbb{Z}^+ & U(k_\circ) = 0 \end{cases} \quad \forall k_\circ = 0, \dots, n \quad (2.3)$$

where $k_\circ = \frac{\beta}{\alpha} - 1$, at t=0 .

It is clear that $k_\circ = 0$ in this case, and by substituting $\beta = \alpha = 1$ the value of $U(k+\frac{\beta}{\alpha})$ will be $U(k+1)$. Next substituting k values in the obtained equation $\forall k = 0, \dots, n$. One can find the values of $U(k_i+1)$ $\forall i = 0, \dots, n$ which present the transformed series of $U(k_i+1)$,after this depending on the derivations of equation (1.6) in properties (1.6).Taking the inverse transform by using the following relation :$u(t) = \sum_{k=0}^{\infty} U(k)(t - t_\circ)^{\alpha k}$ $t_\circ = 0$ $\alpha = 1$, $u(t) = \sum_{k=0}^{\infty} U(k) t^{\alpha k}$

We get the semi analytic solution for equation (2.1) in series form.

Second case When β is fractional

In this case selecting α must satisfie:

- $\alpha \leq \beta - 1$.
- $\frac{\beta}{\alpha} \in \mathbb{Z}^+$

By the same way we can substitute values of β and α in $U(k+\frac{\beta}{\alpha})$, $k = 0, \dots, n$. and apply the same steps to obtain the transformed initial condition using equation (2.3) . Then, we take k values $\forall k = 0, \dots, n$, to find $U(k_i+\frac{\beta}{\alpha})$ $\forall i = 0, \dots, n$. after this, we take the inverse transform:

$u(t)=\sum_{k=0}^{\infty}U(k)(t-t_{\circ})^{\alpha k}$ $t_{\circ}=0$, α is fractional , $u(t)=\sum_{k=0}^{\infty}U(k)t^{\alpha k}$
to obtain the approximation solution for the original equation (1.9) in series form. Next to illustrate the solution procedure and show the feasibility and efficiency of the GDTM we have applied the method . Next, we solve Volterra's population equation using GDTM in two different cases.

(3) Application example

ConsiderL-FVFIDE $D_c^{\beta}u(x)=2e^x-1+\int_0^x u(t)dt+\int_0^1 u(t)dt$ (3.1)

With initial condition u(0)=0 And the exact solution given in [1] as $u(t)=te^t$

To solve the equation (3.1) by using GDTM technique , and the properties given in (1.6) we get transformed equation below :

$\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)}U\left(k+\frac{\beta}{\alpha}\right)=\frac{2}{k!}-\delta(k)+\frac{1}{\alpha k}U\left(k-\frac{1}{\alpha}\right)+\delta(k)\frac{1}{\alpha k}U(k-\frac{1}{\alpha})$ With $k\geq\frac{1}{\alpha}$

$U\left(k+\frac{\beta}{\alpha}\right)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}[\frac{2}{k!}-\delta(k)+\frac{1}{\alpha k}U(k-\frac{1}{\alpha})+\delta(k)\frac{1}{\alpha k}U(k-\frac{1}{\alpha})]$ (3.1.a)

First case:

put $\alpha=\beta=1$,which means $k_{\circ}=0$ then $U(0)=0$

Then substituting α and β values in equation (3.1.a) we get:

$U(k+1)=\frac{\Gamma k+1}{\Gamma k+2}[\frac{2}{k!}-\delta(k)+\frac{1}{\alpha k}U(k-1)+\frac{\delta(k)}{\alpha k}U(k-1)]$ (3.1.b)

Substituting k values in equation (3.1.b) $\forall\, k=0,1,2,\ldots$

For k=0 then $U(1)=1$, For k=1 then $U(2)=1$,For k=2 then $U(3)=\frac{1}{2!}$

And by the same way we can find that $U(k+1)=\frac{1}{k!}$ $\forall\, k\geq 3$ .Now to get semi analytic solution for equation (3.1) formed in a series form applying the inverse transform of equation (3.1.b): $u(t)=\sum_{k=1}^{\infty}U(k)(t-t_0)^{\alpha k}$ $t_0=0$, $\alpha=1$, $u(t)=\sum_{k=1}^{\infty}U(k)t^k$

$u(t)=[U(0)+U(1)t+U(2)t^2+U(3)t^3+U(4)t^4+U(5)t^5+U(6)t^6+\cdots+U(n)t^n+\ldots$

$$u(t)=[0+t+t^2+\frac{1}{2!}t^3+\frac{1}{3!}t^4+\frac{1}{4!}t^5+\frac{1}{5!}t^6+\cdots+\frac{1}{(n-1)!}t^n+\ldots$$

$$u(t)=t[1+t+\frac{1}{2!}t^2+\frac{1}{3!}t^3+\frac{1}{4!}t^4+\frac{1}{5!}t^5+\cdots+\frac{1}{(n)!}t^n+\ldots$$

Then $u(t)=te^t$ and this is exact solution.

Second case: For this case one can select the value of β as: $\beta = 0.5 = \frac{1}{2}$ and $\alpha = 0.25 = 1/4$

To find $k_\circ$: $k_\circ = \frac{\beta}{\alpha} - 1$ Which means $k_\circ = 0,1$ then U(0) =0 , U(1) =0

By substituting β and α values in equation (3.1a) we get

$$U(k+\frac{\frac{1}{2}}{\frac{1}{4}}) = \frac{\Gamma(\frac{1}{4}k+1)}{\Gamma(\frac{1}{4}k+\frac{1}{2}+1)}[\frac{2}{k!} - \delta(k) + \frac{1}{\frac{1}{4}k}U(k+\frac{1}{\frac{1}{4}}) + \delta(k)\frac{1}{\frac{1}{4}k}U(k-\frac{1}{\frac{1}{4}})]$$

$$U(k+2) = \frac{\Gamma\left(\frac{k}{4}+1\right)}{\Gamma\left(\frac{k}{4}+\frac{3}{2}\right)}\left(\frac{2}{k!} - \delta(k) + \frac{4}{k}U(k-4) + \delta(k)\frac{4}{k}U(k-4)\right) \qquad (3.1.c)$$

Once again, substituting k values in equation (3.1.c) $\forall k = 0,1,2,\ldots$

For k=0 then $U(2) = \frac{-2}{\sqrt{\pi}}$, For k=1 then $U(3) = 2\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}$

And by the same way we can find U(4),U(5),… $\forall k \geq 2$

Again applying the inverse transform of equation (3.1.c) to get the approximate solution for equation (3.1) formed in series form: $u(t)=\sum_{k=1}^{\infty} U(k)(t-t_0)^{\alpha k}$ $t_0=0$, $\alpha = \frac{1}{4}$,$u(t)=\sum_{k=0}^{\infty} U(k)t^{\frac{1}{4}k}$

Then $u(t)=U(0)t^0 + U(1)t^{\frac{1}{4}} + U(2)t^{(2*\frac{1}{4})} + U(3)t^{\left(3*\frac{1}{4}\right)} + U(4)t^{\left(4*\frac{1}{4}\right)} + U(5)t^{\left(5*\frac{1}{4}\right)} + \cdots +$

$U(n)t^{\left(n*\frac{1}{4}\right)} + \cdots$

$$u(t) = \frac{2\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}t^{\frac{1}{2}} + \frac{1}{2}\sqrt{\pi}t^{\frac{3}{4}} + \frac{\Gamma\left(\frac{7}{4}\right)}{3\Gamma\left(\frac{9}{4}\right)} + \frac{1}{12\Gamma\left(\frac{5}{2}\right)}t^{\frac{5}{4}} + \ldots.$$

Finding the arbitrary value of t for any fractional derivative order β and calculate it:

For $\beta = 0.8$ then $\alpha = 0.2$ .By substituting β and α values in equation (3.1.a) we get

$$U(k+4) = \frac{\Gamma\left(\frac{k}{5}+1\right)}{\Gamma\left(\frac{k}{5}+\frac{4}{5}+1\right)}[\frac{2}{k!} - \delta(k) + \frac{5}{k}U(k-5) + \delta(k)\frac{5}{k}U(k-5)] \qquad (3.1.d)$$

For example take k=0 then: $U(4) = 1.0737$. and by the same way we can find U(5),U(6),…,U(10) $\forall k \geq 1$, applying the inverse transform of equation (3.1.d) to get approximate solution for equation (3.1) formed in a series form: $u(t) = \sum_{k=0}^{\infty} U(k)t^{\alpha k}$ $t_0=0$, $\alpha = \frac{1}{5}$ $u(t) = U(0)\,t^0 + U(1)t^{\frac{1}{5}} + U(2)t^{\frac{2}{5}} + U(3)\,t^{\frac{3}{5}} + U(4)\,t^{\frac{4}{5}} + U(5)\,t +$ $\ldots + U(n)\,t^{\frac{n}{5}} + \cdots$

For t=0.6 , $\beta = 0.8$, $\alpha = 0.2$. we can obtain u(t) = 2.39731180 .

The illustrated value of u(t) colored as red in table (1) below ,one can find the other entries values of table (1)by the same way.

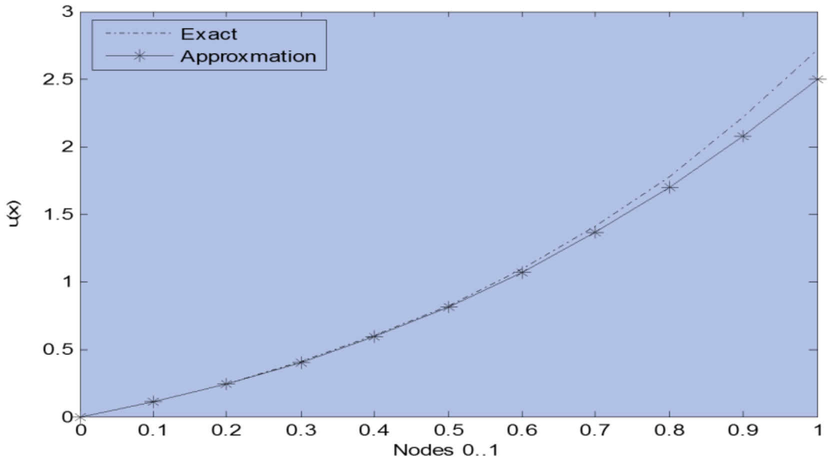

Figure (1):Comparison between the exact solution and approximate solution of the example using GDTM when N=3 ,β=1

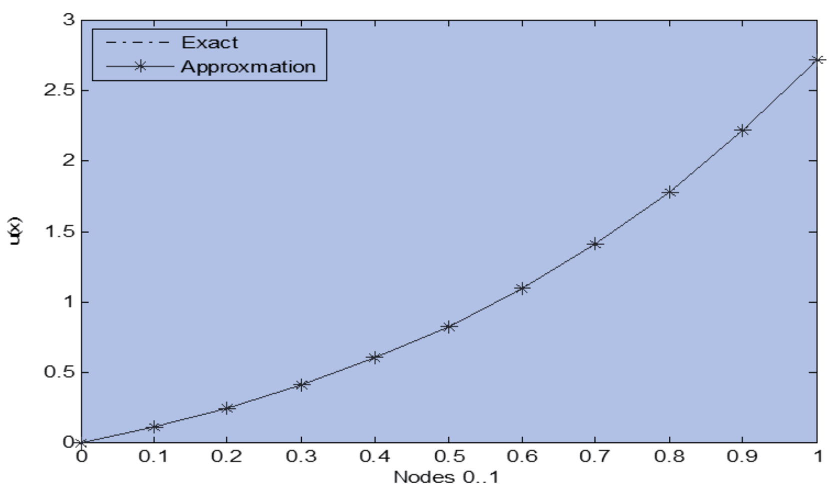

Figure (2):Comparison between the exact solution and approximate solution of the example using GDTM when N=10, β=1

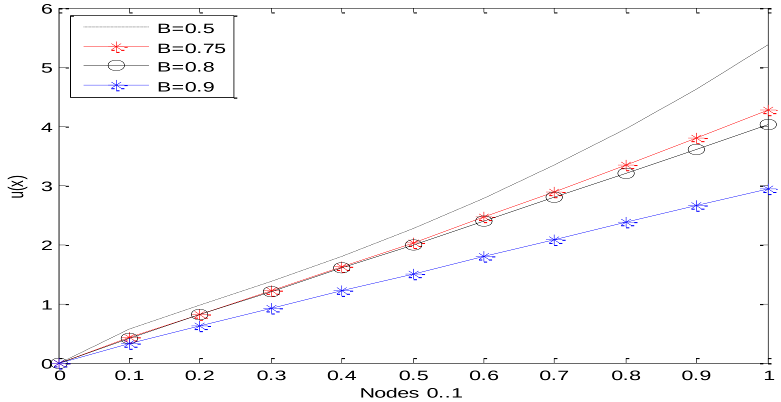

Figure (3):Comparison between the approximate solutions of the example using GDTM when N=10 for different values for the fractional divertive β

Table (1)shows the results of t values for the linear Volterra-Fredholm integro- differential equations of fractional order example solved using GDTM after considering different values for the fractional order divertive β

Values of t	$\alpha = \beta = 1$		$\beta = 0.5$	$\beta = 0.75$	$\beta = 0.8$	$\beta = 0.9$
	N=3	N=10	N=10			
0.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.11050000	0.11051709	0.56714368	0.42801856	0.41569127	0.32116744
0.2	0.24400000	0.24428055	0.96770486	0.82209942	0.81020809	0.62480329
0.3	0.40350000	0.40495764	1.37446591	1.21852312	1.20420909	0.92264607
0.4	0.59200000	0.59672988	1.80629331	1.62210069	1.59972313	1.21689145
0.5	0.81250000	0.82436064	2.27302261	2.03501513	1.99735479	1.50854121
0.6	1.06800000	1.09327128	2.78205461	2.45862051	2.39731180	1.79816938
0.7	1.36150000	1.40962689	3.34001671	2.89390761	2.79965402	2.08614728
0.8	1.69600000	1.78043272	3.95337351	3.34167543	3.20437792	2.37273433
0.9	2.07450000	2.21364271	4.62870685	3.80260817	3.61145109	2.65812164
1.0	2.50000000	2.71828153	5.37286612	4.27731483	4.02082787	2.94245567
s =	0.0534	4.2955e-08				
ans =	0.0070	9.2077e-15				

4. Conclusion:

This method has been successfully applied to find the approximate solution of linear Volterra –Fredholm integro- differential equations of fractional order. Generalized Differential Transformation Method has been employed for first time to solve this type of equations yet its provides more sensible series solutions that converged very rapidly in real physical problems. Also its a straightforward tool that may use to solve many types of integro- differential equations of fractional order that modeled to described physical and engineering real life problems.

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