δ- Best Approximation in 2-Normed Almost Linear Space

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Abstract:

The purpose of this paper is to introduce and discuss the concept of ***δ****-best approximation in 2-normed almost linear space. A concept of* ***δ****-orthogonality in 2- normed almost linear space is also introduced and the relations between these two concepts are obtained in this paper.*

Keywords:
2-normed almost linear space; best approximation in 2- normed almost linear space; δ- best approximation in 2-normed almost linear space ; δ- orthogonalisation in 2- normed almost linear space.

1.Introduction

Diminnie, R. Freese [1] and many others developed new concept like linear 2-normed space.The notion of an almost linear space (als) was introduced by G. Godini [2]-[6]. All spaces involved in this work are over the real field ℝ.S. Gahler, Y.J. Cho, C. S. Elumalai, R. Vijayaragavan [12]-[13] established some characterization of best approximation in terms of 2-semi inner products and normalized duality mapping associated with a linear 2-normed space. Basing on this we introduced a new concept called2-normed almost linear spaceand established some results of best approximation in 2- normed almost linear space[17] andsome results of best simultaneous approximation in 2- normed almost linear space [18] . Mehmet A¸cıkg¨oz [16] introduced the conceptε-Approximation in generalized 2- normed linear space and established some results. Basing on this we introduced a new concept calledδ- Best approximation in 2-normed almost linear space in this paper and we established some results on δ- best approximation and δ- orthogonalisation in 2- normed almost linear space.

2.Preliminaries

Definition: 2.1. Let X be an almost linear space of dimension> 1 and$|||\,.\,|||: X \times X \to \mathbb{R}$ be a real valued function. If ||| . ||| satisfy the following properties

i) $|||\alpha, \beta|||=0$ if and only if α and β are linearly dependent,

ii) $|||\alpha, \beta||| = |||\beta, \alpha|||$,

iii) $|||a\alpha, \beta||| = |a|\ |||\alpha, \beta|||$,

iv) $|||\alpha, \beta-\delta||| \le |||\alpha, \beta-\gamma||| + |||\alpha, \gamma-\delta|||$ for every $\alpha, \beta, \gamma, \delta \in X$ and $a \in \mathbb{R}$.

then $(X,|||.|||)$ is called 2-normed almost linear space. ■

Example: 2.2. Let X = $\mathbb{R}^n$. Let $\alpha, \beta \in X$. Then $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \beta_3, .., \beta_n)$. Define $||| \alpha, \beta ||| = \sqrt{\sum_{i<j} (\alpha_i \beta_j - \beta_i \alpha_j)^2}$ then ||| . ||| satisfies all the properties of 2-normed almost linear space. Hence $(X, |||.|||)$ is 2-normed almost linear space. ■

Example: 2.3.Let $X = \mathbb{R}^n$. Let $\alpha, \beta \in X$. Then $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \beta_3, \ldots\ldots, \beta_n)$. Define $||| \alpha, \beta ||| = \sqrt{\sum_i (\alpha_i - \beta_i)^2}$ then also ||| . ||| satisfies all the properties of 2-normed almost linear space. Hence $(X, |||.|||)$ is 2-normed almost linear space. ■

Example: 2.4. Let (E,|| . ||) be a normed linear space. Let $X = \{ \alpha \in E : \alpha \geq 0\}$.Then X is an als. Define $||| \alpha, \beta ||| = \| \alpha, \beta \|$ Where || . || is 2- norm on E. Then ||| . ||| satisfies all the properties of 2-normed almost linear space. Hence $(X, |||.|||)$ is 2-normed almost linear space. ■

Definition: 2.5.Let X be a 2-normed almost linear space over the real field $\mathbb{R}$ and G a non empty subset of V_X . For a bounded sub set A of X let us define

i) $rad_G(A) = inf_{g \in G} sup_{a \in A} ||| x, a-g |||$ for every $x \in X \backslash V_X$ and 2.1

ii) $cent_G(A) = g_0 \in G : sup_{a \in A} ||| x, a-g_0 ||| = rad_G(A)$ for every $x \in X \backslash V_X$.2.2

The number $rad_G(A)$ is called the chebyshev radius of A with respect to G and an element $g_0 \epsilon\, cent_G(A)$ is called a best simultaneous approximation or chebyshev centre of A with respect to G. ■

Definition: 2.6. When A is a singleton say A= {a}, $a \in X \backslash \bar{G}$ then $rad_G(A)$ is the distance of a to G, denoted by dist(a,G) and defined by

$dist(a,G) = inf_{g \epsilon G} |||x, a-g|||$ for every $x \in X \backslash V_X$ 2.3

and $cent_G(A)$ is the set of all best approximations of 'a' out of G denoted by $P_G(a)$ and defined by $P_G(a) = \{g_0 \in G : |||x, a-g_0||| = dist(a,G), \text{for every } x \in X \backslash V_X\}$ 2.4

Definition: 2.7.Let X be a 2-normed almost linear space. The set G is said to be proximinal if P_G(a) is nonempty for each a$\in X \backslash V_X$.■

It is well known that for any bounded subset A of X we have

i)rad_G(A)=$rad_G(C_0(A))$=$rad_G(\bar{A})$

ii) $cent_G$(A)=$cent_G(C_0(A))$=$cent_G(\bar{A})$

Where $C_0(A)$ stands for the convex hull of A and $\bar{A}$ stands for the closure of A. ■

Definition: 2.8. Let X be a 2-normed almost linear spaces and ϕ≠ G ⊂V_X. We define R$_X$(G) ⊂X in the following way a ∈ R$_X$(G) if for each g∈ G $there\ exists v_g \in V_X$ such that the following conditions are hold

i) ||| x, a-g ||| = ||| x, v_g-g ||| for each $v_g \in V_X$ 2.5

ii) ||| x, a- v ||| ≥ ||| x, v_g- v ||| for every $x \in X \setminus V_X$. 2.6

We have $V_X \subset$R$_X$(G).

If $G_1 \subset G_2$ then R$_X(G_2) \subset$R$_X(G_1)$. ■

3.Main Results

Definition: 3.1 Let X be a 2- normed almost linear spaceover the real field ℝ and G a nonempty subset of V_X and δ > 0. Apoint g_0 ∈ G is said to be δ- best approximation to a ∈A (a bounded subset of X) if |||x, a – g_0 ||| ≤|||x, a – g ||| + δ for all g ∈ G and x∈X\\V_X. ■

Definition: 3.2 For a ∈A, the set of all δ- best approximation to a in A denoted by P_G(a,δ) is defined asP_G(a,δ) = { g_0 ∈G : |||x, a – g_0 ||| ≤|||x, a – g ||| + δ,for all g ∈ G and x∈X\\V_X }.■

Theorem:3.3Let G be a subspace of a 2-normed almost linear space X. Then the set P_G(a,δ) is bounded.

Proof: Letg_1,g_2 ∈ P_G(a,δ) and a ∈ A, then |||x, a – g_1 ||| ≤|||x, a – g ||| + δ and|||x, a – g_2 ||| ≤|||x, a – g ||| + δ,for all g ∈ G and x∈X\\V_X.

Now |||x, g_1 – g_2 ||| = |||x, g_1– a + a – g_2 |||≤|||x, a – g_1 ||| + |||x, a – g_2 |||

≤|||x, a – g ||| + δ + |||x, a – g ||| + δ

≤ 2 |||x, a – g ||| +2δ

$\leq 2\, d(x, G) + 2\delta = M$.

Therefore $|||x, g_1 - g_2||| \leq M$.

Hence $P_G(a,\delta)$ is bounded. ■

Theorem: 3.4 Let G be a subspace of a 2-normed almost linear space X and $a \in A$. Then the set $P_G(a,\delta)$ is convex.

Proof: Let $g_1, g_2 \in P_G(a,\delta)$ and $0 \leq \lambda \leq 1$, then $|||x, a - g_1||| \leq |||x, a - g||| + \delta$ and $|||x, a - g_2||| \leq |||x, a - g||| + \delta$, for all $g \in G$ and $x \in X \backslash V_X$.

$|||x, a - \{\lambda g_1 + (1-\lambda) g_2\}||| = |||x, a - \lambda g_1 - g_2 + \lambda g_2|||$

$= |||x, a - \lambda g_1 - g_2 + \lambda g_2 + \lambda a - \lambda a|||$

$= |||x, \lambda (a - g_1) + (1-\lambda)(a - g_2)|||$

$\leq \lambda (|||x, a - g||| + \delta) + (1-\lambda)(|||x, a - g||| + \delta)$

$\leq |||x, a - g||| + \delta$

This implies $\lambda g_1 + (1-\lambda) g_2 \in P_G(a,\delta)$.

Hence $P_G(a,\delta)$ is convex. ■

Definition: 3.5 Let X be a 2- normed almost linear space, $\delta > 0$ and $a, b \in A$. We call a is δ – orthogonal to b and is denoted by $a \perp_\delta b$ if and only if $|||x, a + \lambda b||| + \delta \geq |||x, a|||$ for all scalars $|\lambda| \leq 1$. ■

For any subsets G_1, G_2 of X, $G_1 \perp_\delta G_2$ if and only if $g_1 \perp_\delta g_2$ for all $g_1 \in G_1$, $g_2 \in G_2$. ■

Theorem: 3.6 Let X be a 2- normed almost linear space and G be a subspace of X and $\delta > 0$. Then for all $a \in A$, $g_0 \in P_G(a,\delta)$ if and only if $(a - g_0) \perp_\delta G$.

Proof: Suppose $g_0 \in P_G(a,\delta)$.

Put $g_1 = g_0 - \lambda g$ for $g \in G$ and $|\lambda| \leq 1$.

Since $g_0 \in P_G(a,\delta)$ and $g_1 \in G$ we have $|||x, a - g_0||| \leq |||x, a - g_1||| + \delta$

$\leq |||x, a - (g_0 - \lambda g)||| + \delta$

$\leq |||x, a - g_0 + \lambda g)||| + \delta$

This implies $(a - g_0) \perp_\delta G$.

Conversely let $(a - g_0) \perp_\delta G$, then $\|x, a - g_0\| \leq \|x, a - g_0 + \lambda g_1\| + \delta$ for all $|\lambda| \leq 1$ and $g_1 \in G$.

For any $g \in G$ by putting $g_1 = g_0 - g$ and $\lambda = 1$ the last inequality implies

$\|x, a - g_0\| \leq \|x, a - g\| + \delta$. This implies $g_0 \in P_G(a,\delta)$. ■

Definition: 3.7 Let X be a 2- normed almost linear space and G be a subspace of X and $\delta > 0$. Define $\hat{G}_\delta = \{ a \in A : \|x, a\| \leq \|x, a - g\| + \delta$ for every $g \in G\} = \{ a \in A : a \perp_\delta G\}$. ■

Lemma: 3.8 Let G be a subspace of a 2-normed almost linear space X. Then for all $a \in A$ and all $\delta > 0$ we have $g_0 \in P_G(a,\delta)$ if and only if $(a - g_0) \in \hat{G}_\delta$.

Proof: By theorem 3.6 we have $g_0 \in P_G(a,\delta)$ if and only if $(a - g_0) \perp_\delta G$.

By the definition of $\hat{G}_\delta$ we have $(a - g_0) \perp_\delta G$ if and only if $(a - g_0) \in \hat{G}_\delta$.

Now $(a - g_0) \perp_\delta G$ implies $g_0 \in P_G(a,\delta)$ by theorem 3.6.

Therefore $g_0 \in P_G(a,\delta)$ if and only if $(a - g_0) \in \hat{G}_\delta$. ■

Theorem: 3.9 Let G be a subspace of a 2-normed almost linear space X, $\delta > 0$ and $\delta \geq \lambda$. Then $\hat{G} \subseteq \hat{G}_\lambda \subseteq \hat{G}_\delta$ and $\cap \hat{G}_\delta = \hat{G}$ for all $\delta > 0$.

Proof: Let $a \in \hat{G}$ then $\|x, a\| \leq \|x, a - g\|$ for all $g \in G$ and $x \in X \setminus V_X$.

Now $\|x, a\| \leq \|x, a - g\| \leq \|x, a - g\| + \lambda, (\lambda > 0)$.

So we have $a \in \hat{G}_\lambda$.

Hence $\hat{G} \subseteq \hat{G}_\lambda$ 3.1

Let $a \in \hat{G}_\lambda$ then $\|x, a\| \leq \|x, a - g\| + \lambda \leq \|x, a - g\| + \delta, (\delta > 0)$.

This impliesa $\in \hat{G}_\delta$.

Therefore $\hat{G}_\lambda \subseteq \hat{G}_\delta$ 3.2

From 3.1 and 3.2 we have $\hat{G} \subseteq \hat{G}_\lambda \subseteq \hat{G}_\delta$. 3.3

Now we prove that $\cap \hat{G}_\delta = \hat{G}$ for all $\delta > 0$.

From 3.3 we have $\hat{G} \subseteq \cap \hat{G}_\delta$ for all $\delta > 0$ 3.4

Let a $\in \cap \hat{G}_\delta$ for all $\delta > 0$.

Then for all $\delta > 0, 0 \leq \||x, a\|| \leq \||x, a - g\|| + \delta$ for all $g \in G$ and $x \in X \setminus V_X$.

Now $0 \leq \||x, a\|| \leq \||x, a - g\|| + \frac{1}{n}$ for every $n \in N$, for all $g \in G$ and $x \in X \setminus V_X$

As $n \to \infty$, $\||x, a\|| \leq \||x, a - g\||$, for all $g \in G$ and $x \in X \setminus V_X$.

Therefore a $\in \hat{G}$.

Hence $\cap \hat{G}_\delta \subseteq \hat{G}$, for all $\delta > 0$ 3.5

From 3.4 and 3.5 we have $\cap \hat{G}_\delta = \hat{G}$ for all $\delta > 0$. ■

Theorem: 3.10 Let G be a subspace of a 2-normed almost linear space X. Then i) If $\delta > 0$, $a \in A$ and $g \in G$ and $a \perp_\delta g$ then $a \perp_\varepsilon g$ for all $\varepsilon \geq \delta$ and ii) If $a \perp_\delta g$ and $|\lambda| < 1$ then $\lambda a \perp_\delta \lambda g$.

Proof: i) Let $\delta > 0$, $a \in A$ and $g \in G$ and $a \perp_\delta g$ then by definition 3.5 we have

$\||x, a\|| \leq \||x, a + \lambda g\|| + \delta$ when $|\lambda| \leq 1$ and $\delta > 0$.

Then $\||x, a\|| \leq \||x, a + \lambda g\|| + \delta \leq \||x, a + \lambda g\|| + \varepsilon$ (since $\varepsilon \geq \delta$).

Therefore $a \perp_\varepsilon g$.

ii) Let $a \perp_\delta g$ and $|\Upsilon| < 1$ then $\||x, a\|| \leq \||x, a + \Upsilon g\|| + \delta$ 3.6

Multiplying both sides of 3.6 by $|\lambda|$ we get
$|\lambda| \, \||x, a\|| \leq |\lambda| \, \||x, a + \Upsilon g\|| + |\lambda| \delta \leq \||\lambda x, \lambda a + \lambda \Upsilon g\|| + |\lambda| \delta$

$\leq \||\lambda x, \lambda a + \mu g\|| + \delta$ and so $\||\lambda x, \lambda a\|| \leq \||\lambda x, \lambda a + \mu g\|| + \delta$.

Therefore $\lambda a \perp_\delta \lambda g$. ■

Theorem: 3.11Let G be a subspace of a 2-normed almost linear space X.
If $\delta > 0$, $a \in A$ and $\varepsilon \geq \delta$ then $P_G(a,\delta) \subseteq P_G(a,\varepsilon)$.

Proof: Let $g_0 \in P_G(a,\delta)$ then by definition 3.1 we have
$\||x, a - g_0\|| \leq \||x, a - g\|| + \delta$ for all $g \in G$ and $x \in X \setminus V_X$ and $\delta > 0$.

Then $\||x, a - g_0\|| \leq \||x, a - g\|| + \delta$

$\leq \||x, a - g\|| + \varepsilon$ (since $\varepsilon > \delta$)

Therefore $g_0 \in P_G(a,\varepsilon)$.

Hence $P_G(a,\delta) \subseteq P_G(a,\varepsilon)$. ■

Reference

[1]	R.Freese and S.Gähler	:	Remarks on semi 2-normed space, Math. Nachr. 105(1982) 151-161.
[2]	G. Godini	:	An approach to generalizing Banach spaces. Normed almost linear spaces, Proceedings of the 12th winter school on Abstract Analysis.5 (1984) 33-50.
[3]	G. Godini	:	Best approximation in normed almost linear spaces. In constructive theory of functions. Proceedings of the International conference on constructive theory of functions. (Varna 1984) Publ. Bulgarium Academy of sciences; Sofia (1984) 356-363.
[4]	G. Godini	:	A Frame work for best simultaneous approximation: Normed almost linear spaces. J.Approxi. Theory. 43 (1985) 338-358.
[5]	G. Godini	:	On Normed almost linear spaces, Math.Ann.279 (1988) 449-455.
[6]	G. Godini	:	Operators in Normed almost linear spaces Proceedings of the 14th winter school on Abstract Analysis (Srni.1986).Suppl. Rend. Circ. Mat. Palermo II. Numero. 14(1987) 309-328.
[7]	Sung Mo Im and Sang Han Lee	:	Uniqueness of basis for almost linear space, Bll. korean Math.Soc.34 (1997) 123-133.
[8]	S.Elumalai,Y.J.Cho and S.S. Kim	:	Best approximation sets in linear 2- normed space Comm.Korean Math. Soc. 12 (1997) 619-629

[9]	Geetha S.Rao and T.L.Bhaskeramurthi	:	Chebyshev Centers Proximinality and Farthest points in Strong normed almost linear spaces, Approx.Theory & its Appli.13 (1997) 99-111.
[10]	S.S.Dragomir	:	Some characterization of bestapproximation in normed linear spaces. Acta Mathematical Vietnamica. 25 (2000) 359-366.
[11]	S. Elumalai and R. Vijayaragavan	:	Best simultaneous approximation in linear 2-normed spaces .General Mathematics.16 (2008) 73-81.
[12]	S. Elumalai and R. Vijayaragavan	:	Characterization of best approximation in linear 2-normed spaces. General Mathematics. 17 (2009) 141-160.
[13]	Mehmet A¸cıkg¨oz		ε-Approximation in generalized 2- normed spaces MATEMATIQKI VESNIK61 (2009), 159–163
[14]	A. Khorasani and M.A. Moghaddam	:	Best approximation in Probabilistic 2-normed Spaces, Novi SadJ.Math. 40 (2010) 103-110.
[15]	Y. Dominic and M. Marudai	:	Best approximation in uniformly convex 2- normed spaces, Int. journal of Math. Analysis.6 (2012) 1015-1021.
[16]	T.Markandeya Naidu & Dr. D. Bharathi	:	Best approximation in2- normed almost linear space“International Journal of Engineering Research & Technology” 2 (2013)3569-3573
[17]	T.Markandeya Naidu & Dr. D. Bharathi	:	Best simultaneous approximation in2- normed almost linearspace“International Journal of EngineeringResearch & Technology” 3(2014)1920-1925