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Integral Transformation and Different Forms

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ABSTRACT

The main problem under study is the construction of the complete convergent series development of integral transforms. The proposed paper is a study on integral transformation and its different forms. It also studies the relation between integral transformations and different forms. Integral transform, mathematical operator that produces a new function $f(y)$ by integrating the product of an existing function $F(x)$ and a so-called kernel function $K(x, y)$ between suitable limits. The process, which is called transformation, is symbolized by the equation $f(y) = \int K(x, \ y)F(x)dx$. Several transforms are commonly named for the mathematicians who introduced them: in the Laplace transform, the kernel is e^{-xy} and the limits of integration are zero and plus infinity; in the Fourier transform, the kernel is $(2\pi)^{-1/2}e^{-ixy}$ and the limits are minus and plus infinity.

INTEGRAL TRANSFORMATION:

Integral transform, mathematical operator that produces a new function $f(y)$ by integrating the product of an existing function $F(x)$ and a so-called kernel function $K(x, y)$ between suitable limits. The process, which is called transformation, is symbolized by the equation $f(y) = \int K(x, \ y)F(x)dx$. Several transforms are commonly named for the mathematicians who introduced them: in the Laplace transform, the kernel is e^{-xy} and the limits of integration are zero and plus infinity; in the Fourier transform, the kernel is $(2\pi)^{-1/2}e^{-ixy}$ and the limits are minus and plus infinity.

Integral transforms are valuable for the simplification that they bring about, most often in dealing with differential equations subject to particular boundary conditions. Proper choice of the class of transformation usually makes it possible to convert not only the derivatives in an intractable differential equation but also the boundary values into terms of an algebraic equation that can be easily solved. The solution obtained is, of course, the transform of the solution of the original differential equation, and it is necessary to invert this transform to complete the operation. For the common transformations, tables are available that list many functions and their transforms.

The precursor of the transforms were the Fourier series to express functions in finite intervals. Later the Fourier transform was developed to remove the requirement of finite intervals.Using the Fourier series, just about any practical function of time (the voltage across the terminals of an electronic device for example) can be represented as a sum of sines and cosines, each suitably scaled (multiplied by a constant factor), shifted (advanced or

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retarded in time) and "squeezed" or "stretched" (increasing or decreasing the frequency). The sines and cosines in the Fourier series are an example of an orthonormal basis (Doetsch, G. (2012)).

DERIVATION:

If you have a function f(x) and a function k(x,s) then you can (as long as the product of f(x) times k(x,s) is integrable on the set X) always form another function of a new variable s as follows:

$$F(s) = \int_X k(x,s) f(x) dx$$

We have just "transformed" the function f(x) into the function F(s) via an "integral transform." Why the hell would anyone want to do this? Well, the function F(s) is sometimes easier to work with than f(x) itself, or tells us interesting information about f(x) that it would be hard to figure out in other ways.

Of course, the interpretation of this new function F(s) will depend on what the function k(x,s) is. Choosing k(x,s) = 0, for example, will mean that F(s) will always be zero. This is pretty boring and tells us nothing about f(x).

Whereas choosing $k(x,s) = x^s$ will give us the sth moment of f(x) whenever f(x) is a probability density function. For s=1 this is just the mean of the distribution f(x). Moments can be really handy.

A particularly interesting class of functions k(x,s) are ones that produce invertible transformations (which implies that the transform destroys no information contained in the original function) (Bracewell, R. (1965)). This will occur when there exists a function K(x,s) (the inverse of k(x,s)) and a set S such that

$$f(x) = \int_S K(x,s) F(s) ds$$

That undoes the original transformation (or, at least, undoes it for some large class of functions f(x)).

Whenever this is the case, we can view our operation as changing the domain from x space to s space. Each function f of x becomes a function F of s that we can convert back to f later if we so choose to. Hence, we're getting a new way of looking at our original function!

It turns out that the Fourier transform, which is one of the most useful and magical of all integral transforms, is invertible for a large class of functions. We can construct this transformation by setting:

$$k(x,s) = e^{-ixs}$$
$$K(x,s) = e^{ixs}$$

which leads to a very nice interpretation for the variable s. We call F(s) in this case the "Fourier transform of f", and we call s the "frequency". Why is s frequency? Well, we have Euler's famous formula:

$$e^{ixs} = \cos(xs) + i\sin(xs)$$

So modifying s modifies the oscillatory frequency of cos(xs) and sin(xs) and therefore of k(x,s). There is another reason to call s frequency though. If x is time, then f(x) can be thought of as a waveform in time, and in this case |F(s)| happens to represent the strength of the frequency s in the original signal. You know those bars that bounce up and down on stereo systems? They take the waveforms of your music, which we call f(x), then apply (a discrete version of) the Fourier transform to produce F(s). They then display for you (what amounts to) the strength of these frequencies in the original sound, which is |F(s)|. This is essentially like telling you how strong different notes are in the music sound wave (Tranter, C. J. (1951)).

Below are a few other neat examples of integral transform.

The Laplace transform:

$$k(x,s) = e^{-xs}$$

This is handy for making certain differential equations easy to solve (just apply this transformation to both sides of your equation)

The Hilbert transform:

$$k(x,s) = \frac{1}{\pi}\frac{1}{x-s}$$

This has the property that (under certain conditions) it transforms a harmonic function into its harmonic conjugate, elucidating the relationship between harmonic functions and holomorphic functions, and therefore connecting problems in the plane with problems in complex analysis.

The identity transforms:

$$k(x,s) = \delta(x-s)$$

Here δ is the dirac delta function. This is the transformation that leaves a function unchanged, and yet it manages to be damn useful.

EXAMPLES:

In this example, polynomials in the complex frequency domain (typically occurring in the denominator) correspond to power series in the time domain, while axial shifts in the complex frequency domain correspond to damping by decaying exponentials in the time domain (Debnath, L., & Bhatta, D. (2014)).

The Laplace transform finds wide application in physics and particularly in electrical engineering, where the characteristic equations that describe the behavior of an electric circuit in the complex frequency domain correspond to linear combinations of exponentially damped, scaled, and time-shifted sinusoids in the time domain. Other integral transforms find special applicability within other scientific and mathematical disciplines.

Another usage example is the kernel in path integral:

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(x',t')K(x,t;x',t')dx'.$$

This state that the total amplitude to arrive at (x,t) [that is, $\psi(x,t)$] is the sum, or the integral, over all possible value of x' of the total amplitude to arrive at the point (x',t') [that is, $\psi(x',t')$] multiplied by the amplitude to go from x' to x [that is, $K(x,t;x',t')$.[1] It is often referred to as the propagator of a given system. This (physics) kernel is the kernel of integral transform. However, for each quantum system, there is a different kernel.

LAPLACE TRANSFORMATION:

Let f(t) be a given function which is defined for all positive values of t, if (Davies, B. (2012))

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

exists, then F(s) is called *Laplace transform* of f(t) and is denoted by

$L\{f(t)\} = F(s) = \quad e^{-st} f(t)\, dt$

The inverse transform, or inverse of L{f(t)} or F(s), is

$f(t) = L^{-1}\{F(s)\}$

Where *s is real or complex* value.

[Examples]

$$L\{1\} = \frac{1}{s} \quad ; \qquad L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{\cos wt\} = \int_0^{\infty} e^{-st} \cos wt\, dt$$

$$= \frac{e^{-st}(-s\cos wt + w\sin wt)}{w^2+s^2}\Bigg|_{t=0}^{\infty}$$

$$= \frac{s}{s^2+w^2}$$

(Note that s 0, otherwise $e^{-st}\big|_{t=\infty}$ diverges)

$L\{\sin wt\} = \int_0^\infty e^{-st}\sin wt\,dt$ (integration by parts)

$$= \frac{-e^{-st}\sin wt}{s}\Bigg|_{t=0}^{\infty} + \frac{w}{s}\int_0^\infty e^{-st}\cos st\,dt$$

$$= \frac{w}{s}\int_0^\infty e^{-st}\cos st\,dt$$

$$= \frac{w}{s} L\{\cos t\} = \frac{w}{s^2+w^2}$$

Note that

$$L\{\cos wt\} = \int_0^\infty e^{-st}\cos wt\,dt$$

(integration by parts)

$$= \frac{-e^{-st}\cos wt}{s}\Bigg|_{t=0}^{\infty} - \frac{w}{s}\int_0^\infty e^{-st}\sin wt\,dt$$

$$= \frac{1}{s}\quad\frac{w}{s} L\{\sin wt\}$$

$$\Rightarrow L\{\sin wt\} = \frac{w}{s} L\{\cos wt\} = \frac{w}{s^2}\quad\frac{w^2}{s^2} L\{\sin wt\}$$

$$\Rightarrow L\{\sin wt\} = \frac{w}{s^2+w^2}$$

$$L\{t^n\} = \int_0^\infty t^n e^{-st}\,dt \text{(let } t = z/s, dt = dz/s\text{)}$$

$$= \int_0^\infty \left[\frac{z}{s}\right]^n e^{-z}\frac{dz}{s} = \frac{1}{s^{n+1}}\int_0^\infty z^n e^{-z}\,dz$$

$$= \frac{G(n+1)}{s^{n+1}} \quad \text{(Recall } f(x) = \int_0^\infty e^{-t}t^{x-1}\,dt\text{)}$$

If n = 1, 2, 3, . . . n(n+1) = n!

$\Rightarrow \quad L\{t^n\} = \frac{n!}{s^{n+1}}$ where n is a positive integer

[Theorem]Linearity of the Laplace Transform

$L\{af(t)+b\,g(t)\} = a\,L\{f(t)\} + b\,L\{g(t)\}$ where a and b are constants.

[Exampl**e]** $\quad L\{e^{at}\} = \frac{1}{s-a}$

$L\{\sinh at\} = L\{e^{at}/2\,(1-e^{-2at})\}$

Since

$$L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}L\{e^{at}\}\quad\frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2-a^2}$$

FOURIER INTEGRAL:

To begin, we let f be a nonperiodic function with domain $(-\infty,\infty)$ and such that

(a) On every finite interval, f satisfies the Dirichlet conditions (Bracewell, R. (1965)).

(b)The improper integral $\int_{-\infty}^{\infty}|f(t)|\,dt$ exists.

We then define a periodic function f_p of period 2p in terms of f as follows:

$$f_p = \begin{cases} f(t) & -p \le t \le p \\ f_p(t+2p) & t \text{ real} \end{cases} \quad (1)$$

Clearly, f(t) is the limit $f_p(t)$ as $p \to \infty$. Because f satisfies the Dirichlet conditions on every finite interval, it follows that for every value of p the function f_p satisfies the same conditions in each period and hence possesses a valid Fourier series. Then we can expand f_p in the complex exponential form and we can therefore write

(2) $$f_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{ni\pi t}{p}\right)$$

where $$c_n = \frac{1}{2p}\int_{-p}^{p} f_p(t)\exp\left(-\frac{ni\pi t}{p}\right) dt$$

Substituting c_n into the expression for $f_p(t)$ gives

(3)
$$f_p(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2p}\int_{-p}^{p} f_p(t)\exp\left(-\frac{ni\pi t}{p}\right) dt\right]\exp\left(\frac{ni\pi t}{p}\right)$$
$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-p}^{p} f_p(t)\exp\left(-\frac{ni\pi t}{p}\right) dt\right]\exp\left(\frac{ni\pi t}{p}\right)\frac{\pi}{p}$$

Now, let us denote the **frequency** of the general term by $\omega_n = \frac{n\pi}{p}$ and the difference in frequency between successive terms by $\Delta\omega = \frac{\pi}{p}$.

Then

(4)
$$f_p(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-p}^{p} f_p(t)\exp\left(-\frac{i\omega_n t}{p}\right) dt\right]\exp\left(\frac{i\omega_n t}{p}\right)\Delta\omega$$

If we now set

(5) $$C_p(\omega) = \frac{1}{2\pi}\int_{-p}^{p} f_p(t)\exp\left(-\frac{i\omega_n t}{p}\right) dt$$

and for each p define a function F_p by

(6) $F_p(\omega) = C_p(\omega)\exp(i\omega t)$

Eq.(6) becomes simply

(7) $$f_p(t) = \sum_{n=-\infty}^{\infty} F_p(\omega_n)\Delta\omega$$

In the limiting case, as $p \to \infty$ and $\Delta\omega \to 0$, we have $f_p \to f$ and

(8) $$C(\omega) = \lim_{p\to\infty} C_p(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\, dt$$

and

(9)
$$F(\omega) = \lim_{p\to\infty} F_p(\omega) = \lim_{p\to\infty} C_p(\omega)\exp(i\omega t) = C(\omega)\exp(i\omega t)$$

the nonperiodic limit f(t) of $f_p(t)$ is correctly given by the formula

(10)
$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, d\omega = \int_{-\infty}^{\infty} C(\omega)\exp(i\omega t)\, d\omega$$
$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\, dt\right]\exp(i\omega t)\, d\omega$$

Therefore we do have the following theorem:-

Theorem

If on every finite interval, f satisfies the Dirichlet conditions and if the improper integral $\int_{-\infty}^{\infty} |f(t)|\, dt$ exists, the **Fourier integral**

$$f(t) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\, dt\right]\exp(i\omega t)\, d\omega$$

(2.1)

gives the value f at every point where f is continuous.

The Fourier integral (20.1) can be written as the integral

$$f(t) = \int_{-\infty}^{\infty} C(\omega)\exp(i\omega t)\, d\omega \quad (2.2)$$

in which C is the **coefficient function**

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\, dt \quad (2.3)$$

Fourier Integral Representation

Just as in the case in Fourier series, the above Fourier integral can be written in alternative form. To do this, we change the dummy variable of integration from t to □ in the inner integral in (2.1) and then move $e^{i□t}$ across the integral sign, which we can do because it does not involve . This gives

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)\exp(-i\,\omega(\tau-t))d\tau\,d\omega \quad (2.4)$$

In this, we can replace the exponential by its trigonometric equivalent, getting

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)[\cos\omega(\tau-t)-i\sin\omega(\tau-t)]d\tau\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)\cos\omega(\tau-t)\,d\tau\,d\omega$$

$$-\frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)\sin\omega(\tau-t)\,d\tau\,d\omega \quad (2.5)$$

If the function f is purely real, the second term of the R.H.S. will vanish. This gives

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)\cos\omega(\tau-t)\,d\tau\,d\omega \quad (2.6)$$

Since the integral of Eq.(20.6) is an even function of,we need perform the integration onlybetween 0 and ∞, provided we multiply the result by 2. This gives us the modified form

$$f(t)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(\tau)\cos\omega(\tau-t)\,d\tau\,d\omega \quad (2.7)$$

which, when cos(t - t) is expanded, becomes

$$f(t)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(\tau)[\cos\omega\tau\cos\omega t+\sin\omega\tau\sin\omega t]d\tau\,d\omega$$

$$=\int_{0}^{\infty}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\cos\omega\tau\,d\tau\right\}\cos\omega t\,d\omega+\int_{0}^{\infty}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\sin\omega\tau\,d\tau\right\}\sin\omega t\,d\omega \quad (2.8)$$

The two integrals in R.H.S. of the above equality are called the **coefficient functions**.

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos\omega t\,dt$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin\omega t\,dt \quad (2.9)$$

Hence we arrive at the **standard Fourier integral representation** of f

$$f(t)=\int_{0}^{\infty}A(\omega)\cos\omega t\,d\omega+\int_{0}^{\infty}B(\omega)\sin\omega t\,d\omega \quad (2.10)$$

If f is an even function, $B(\omega) = 0$

$$f(t)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(\tau)\cos\omega\tau\cos\omega t\,d\tau\,d\omega$$

f is even (2.11)

This is so-called **Fourier cosine integral** of f.

If f is an odd function, $A(\omega) = 0$

$$f(t)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(\tau)\sin\omega\tau\sin\omega t\,d\tau\,d\omega$$

f is odd (2.12)

This is so-called

Fourier sine integral of f.

Example 5.17 Find f(t) such that

$$f(t) = 2t^2+\int_0^t f(t-u)\,e^{-u}\,du$$

Solution : It is clear that

$$f(t)*g(t)=\int_0^t f(t-u)\,e^{-u}\,du$$

and by Theorem 5.7.

$$L\{f(t)*g(t)\}=L\{f(t)\}L\{g(t)\} = F(s)\,\frac{1}{s+1}$$

By taking the Laplace transform of both sides of the integral equation we get

$$L\{f(t)\} = L\{2t^2\} + F(s)\,\frac{1}{s+1}$$

$$\text{or } F(s) = 2.\,\frac{2}{s^3}+\frac{1}{s+1}F(s)$$

$$\text{or } F(s)\left[\frac{s}{s+1}\right]=\frac{4}{s^3}$$

$$\text{or } F(s) = \frac{4(s+1)}{s^4}=\frac{4}{s^3}+\frac{4}{s^4}$$

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Taking inverse Laplace transform we get

$$f(t) = 2\,L^{-1}\left\{\frac{2}{s^3}\right\} + \frac{2}{3}L^{-1}\left(\frac{3!}{s^4}\right) = 2t^2 + \frac{2}{3}t^3$$

Example 5.18 Find the function f(t) if

$$f(t) = t + \int_0^t f(u)\sin(t-u)du$$

Solution: We can identify the integral as f(s)∗h(t) where h(t) = sin t

Taking the Laplace transform of both sides of the integral equation we get

L {f(t)} = L {t} + L {f(t)∗h(t)}.

By Theorem 5.7

L{f(t)∗h(t)}= L{f(t)} L {h(t)}

$$= F(s)\,\frac{1}{s^2+1}$$

Thus

$$F(s) = \frac{1}{s^2} + F(s)\,\frac{1}{s^2+1}$$

$$\text{or}\quad F(s)\,\frac{s^2}{s^2+1} = \frac{1}{s^2}$$

$$\text{or}\quad F(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

Taking the inverse Laplace transform of this equation we get

$$f(t) = t + \frac{1}{6}t^3$$

Example 5.19 A spring is attached to a 16-lb block resting on a frictionless plane. A horizontal force of 4 lb is applied to the block through the spring for 3 sec. and then released. Describe the resulting motion if the block is initially at rest and the spring constant is equal to 2.

Solution: The differential equation of the system is

$$\frac{16}{32}\,y''(t) + 2\,y(t) = f(t),\ y(0)=0,\ y'(0)=0$$

where the applied force f(t) is given by

$$f(t) = \begin{cases} 4, & 0<t<3 \\ 0, & t>3 \end{cases}$$

The **unit step function** is denoted by $u_a(t)$ and defined by

$$u_a(t) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$

$$f(t) = 4-4\,u_3(t)$$

The differential equation can be written as

$$\frac{1}{2}\,y''(t) + 2y(t) = 4 - 4\,u_3(t)$$

$$\text{or}\qquad y''(t) + 4y(t) = 8-8\,u_3(t)$$

Let L {y(t)} = Y(s) then

$$s^2\,Y(s) + 4Y(s) = \frac{8}{s} - \frac{8}{s}e^{-3s}$$

(See theorem 5.3)

Therefore

$$Y(s) = \frac{8}{s(s^2+4)} - \frac{8}{s(s^2+4)}e^{-3s}$$

Using partial fractions expansions,

$$Y(s) = \frac{2}{s} - \frac{2s}{s^2+4} - \frac{2}{s}e^{-3s} + \frac{2s}{s^2+4}e^{-3s}$$

The appropriate inverse Laplace transforms yield

$$y(t) = 2-2\cos 2t - 2\,u_3(t) + 2\cos 2(t-3)\,u_3(t)$$

$$= 2(1-\cos 2t) - 2\,[1-\cos 2(t-3)]\,u_3(t)$$

$$= \begin{cases} 2(1-\cos 2t) & ,0<t<3 \\ 2[\cos 2(t-3)-\cos 2t], & t>3 \end{cases}$$

Example 5.20 Solve the problem

y''-2y'-8y=f(t); y0)=1,y'(0)=0.

Solution : Apply the Laplace transform inserting the initial values, to obtain

$$L(y''-2y'-8y) = (s^2Y(s)-s) - 2(sY(s)-1) - Y(s) = F(s).$$

Then

$$(s^2-2s-8)Y(s) - s + 2 = F(s).$$

So

$$Y(s) = \frac{1}{s^2-2s-8}F(s) + \frac{s-2}{s^2-2s-8}$$

Use a partial fractions decomposition to write

$$Y(s) = \frac{1}{6}\,\frac{1}{s-4}F(s) - \frac{1}{6}\,\frac{1}{s+2}F(s) + \frac{1}{3}\,\frac{1}{s-4}$$

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$$+\frac{2}{3}\frac{1}{s+2}$$

Taking inverse Laplace transform, we get

$$y(t)=\frac{1}{6}e^{4t}*f(t)-\frac{1}{6}e^{-2t}*f(t)+\frac{1}{3}e^{4t}+\frac{2}{3}e^{-2t}$$

This is the solution, for any function f having a convolution with e^{4t} and e^{-2t}

Laplace Transform Solution of Systems

The analysis of mechanical and electrical systems having several components can lead to systems of differential equations that can be solved using the Laplace transform.

Example 5.21 Consider the system of differential equations and initial conditions for the functions x and y:

$$x'' - 2x' + 3y' + 2y = 4.$$

$$2y' - x' + 3y = 0.$$

$x(0) = x'(0)=y(0)=0$. Solve for x and y.

By applying the Laplace transform to the differential equations, incorporating the initial conditions, we get

$$s^2X(s)-2sX(s)+3sY(s)+2Y(s)=\frac{4}{s}$$

$$2sY(s)-X(s)+3Y(s)=0.$$

Solve these equations for X(s) and Y(s) to get

$$X(s)=\frac{4s+6}{s^2(s+2)(s-1)} \text{ and}$$

$$Y(s)=\frac{2}{s\ (s+2)(s-1)}$$

A partial fractions decomposition yields

$$X(s)=-\frac{7}{2}\frac{1}{s}-3\frac{1}{s^2}+\frac{1}{6}\frac{1}{s+2}+\frac{10}{3}\frac{1}{s-1}$$

and

$$Y(s)=-\frac{1}{s}+\frac{1}{3}\frac{1}{s+2}+\frac{2}{3}\frac{1}{s-1}.$$

Applying the inverse Laplace transform, we obtain the solution

$$x(t)=-\frac{7}{2}-3t+\frac{1}{6}e^{-2t}+\frac{10}{3}e^{t}$$

and

$$y(t)=-1+\frac{1}{3}e^{-2t}+\frac{2}{3}e^{t}.$$

Example 5.22 Consider the spring/mass system of Figure 5.6. Let $x_1 = x_2 = 0$ at the equilibrium position, where the weights are at rest. Choose the direction to the right as positive and suppose the weights are at positions $x_1(t)$ and $x_2(t)$ at time t.

By two applications of Hooke's law, the restoring force on m_1 is

$$-k_1x_1+k_2(x_2-x_1)$$

and that on m_2 is

$$-k_2(x_2-x_1)-k_3x_2.$$

By Newton's second law of motion,

$$m_1x''_1=-(k_1+k_2)x_1+k_2x_2+f_1(t).$$

and

$$m_2x''_2=k_2x_1-(k_2+k_3)\ x_2+f_2(t)$$

These equations assume that damping is negligible but allow for forcing functions acting $f_1(t)$ and $f_2(t)$ on each mass.

As a specific example, suppose $m_1 = m_2 = 1$ and $k_1 = k_3 = 4$ while $k_2 = \frac{5}{2}$. Suppose $f_2(t) = 0$, so no external driving force acts on the second mass, while a force of magnitude $f_1(t) = 2[1-H(t-3)]$ acts on the first. This hits the first mass with a force of constant magnitude 2 for the first 3 seconds, then turns off. Now the system of equations for displacement functions is

$$x''_1=-\frac{13}{2}x_1+\frac{5}{2}x_2+2[1-H(t-3)],$$

$$x''_2=\frac{5}{2}x_1-\frac{13}{2}x_2.$$

If the masses are initially at rest at the equilibrium position, then

$$x_1(0)=x_2(0)=x'_1(0)=x'_2(0)=0$$

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Apply the Laplace transform to each equation of the system to get

$$s^2X_1 = -\frac{13}{2}X_1 + \frac{5}{2}X_2 + \frac{2(1-e^{-3s})}{s},$$

$$s^2X_2 = \frac{5}{2}X_1 - \frac{13}{2}X_2.$$

Solve these to obtain

$$X_1(s) = \frac{2}{(s^2+9)(s^2+4)}\left(s^2+\frac{13}{2}\right)\frac{1}{s}(1-e^{-3s})$$

And

$$X_2(s) = \frac{5}{(s^2+9)(s^2+4)}\ \frac{1}{s}\ (1-e^{-3s})$$

For applying the inverse Laplace transform, use a partial fractions decomposition to write

$$X_1(s) = \frac{1}{s+6}\ \frac{1}{s+6}\ \frac{1}{4}$$

$$\frac{s}{s^2+4} - \frac{1}{9}\frac{s}{s^2+9} - \frac{13}{36}\frac{1}{s}e^{-3s} + \frac{1}{4}\frac{s}{s^2+4}e^{-3s} + \frac{1}{9}\frac{s}{s^2+9}e^{-3s}$$

And

$$X_2(s) = \frac{5}{36}\ \frac{1}{s} - \frac{1}{4}$$

$$\frac{s}{s^2+4} + \frac{1}{9}\frac{s}{s^2+9} - \frac{5}{36}\frac{1}{s}e^{-3s} + \frac{1}{4}\frac{s}{s^2+4}e^{-3s} - \frac{1}{9}\frac{s}{s^2+9}e^{-3s}$$

Apply the inverse Laplace transform to obtain the solution:

$$x_1(t) = \frac{13}{36} - \frac{1}{4}\cos(2t) - \frac{1}{9}\cos(3t)$$

$$+$$

$$\left[-\frac{13}{36} + \frac{1}{4}\cos(2(t\text{-}3)) + \frac{1}{9}\cos(3(t\text{-}3))\right]H(t\text{-}3),$$

$$x_2(t) = \frac{5}{36} - \frac{1}{4}\cos(2t) + \frac{1}{9}\cos(3t)$$

$$+$$

$$\left[-\frac{5}{26} + \frac{1}{4}\cos(2(t\text{-}3)) - \frac{1}{9}\cos(3(t\text{-}3))\right]H(t\text{-}3).$$

Example 5.23

In the circuit of suppose the switch is closed at time zero. We want to know the current in each loop. Assume that both loop currents and the charges on the capacitors are initially zero.

Apply Kirchhoff's laws to each loop to get

$$40i_1 + 120(q_1\text{-}q_2) = 10$$

$$60i_2 + 120\, q_2 = 120(q_1\text{-}q_2).$$

Since $i = q'$, we can write $q(t) = \int_0^t i(u)\, du + q(0)$. Put into the two circuit equations, we get

$$40i_1 + 120\int_0^t [i_1(u) - i_2(u)]\, du + 120[q_1(0)\text{-}q_2(0)] = 10$$

$$60i_2 + 120\int_0^t i_2(u)du + 120q_2(0) = 120$$

$$\int_0^t [i_1(u) - i_2(u)]du + 120[q_1(0)\text{-}q_2(0)].$$

Put $q_1(0) = q_2(0) = 0$ in this system to get

$$40i_1 + 120\int_0^t [i_1(u) - i_2(u)]\, du = 10$$

$$60i_2 + 120\int_0^t i_2(u)du = 120\int_0^t [i_1(u) - i_2(u)]du$$

Apply the Laplace transform to each equation to get

$$40I_1 + \frac{120}{s}I_1 - \frac{120}{s}I_2 = \frac{10}{s}$$

$$60I_1 + \frac{120}{s}I_2 = \frac{120}{s}I_1 - \frac{120}{s}I_2.$$

Rearranging the terms, we get

$$(s+3)\, I_1 - 3I_2 = \frac{1}{4}$$

$$2I_1 - (s+4)\, I_2 = 0.$$

Solve these to get

$$I_1(s) = \frac{s+4}{4(s+1)(s+6)} = \frac{3}{20}\ \frac{1}{s+1} + \frac{1}{10}\ \frac{1}{s+6}$$

and

$$I_2(s) = \frac{1}{2(s+1)(s+6)} = \frac{1}{10}\ \frac{1}{s+1} - \frac{1}{10}\ \frac{1}{s+6}.$$

Now use the inverse Laplace transform to find the solution

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$i_1(t)= \frac{3}{20} e^{-t} + \frac{1}{10} e^{-6t}$, $i_2(t)= \frac{1}{10} e^{-t} - \frac{1}{10} e^{-6t}$.

Example 5.24 Solve the initial-value problem :

$t y'' + (4t – 2)y' – 4y = 0; y (0) = 1.$

If we write this differential equation in the form $y'' + p(t)y' + q(t)y = 0$, then we must choose $p(t) = (4t – 2)/t$, and this is not defined at $t = 0$, where the initial condition is given.

Apply the Laplace transform to the differential equation to get

$L [ty'']+4 L [ty']-2 L [y']-4 L [(y)]=0=0.$

First,

$$L [ty'']=-\frac{d}{ds} L[y'']=-\frac{d}{ds} [s^2Y –sy(0)-y'(0)]$$

$$= - 2sY(s)-s^2Y'(s)+1$$

because $y(0)=1$ and $y'(0)$, though unknown, is constant and has zero derivative. Next,

$$L [ty'] = -\frac{d}{ds} L [y']$$

$$= -\frac{d}{ds} [sY(s) –y(0)]= -Y(s) – sy'(s)$$

Finally,

$$L [y']=sY(s)-y(0) =sY(s)-1$$

The transform of the differential equation is therefore

$-2sY(s)-s^2Y'(s)+1-4Y(s)-4sY'(s)-2sY(s)+2-4Y(s)=0$

Then

$$Y' + \frac{4s+8}{s(s+4)} Y = \frac{3}{s(s+4)}$$

This is a linear first-order differential equation, and we will find an integrating factor. First compute

$$\int \frac{4s+8}{s(s+4)} ds = In [s^2 (s+4)^2].$$

Then

$$e^{In(s^2 (s+4)^2)} = s^2 (s+4)^2$$

is an integrating factor. Multiply the differential equation by this factor to obtain

$s^2(s+4)^2Y'+(4s+8)s(s+4)Y=3s(s+4),$

or

$[s^2(s+4)^2Y]'=3s(s+4).$

Integrate to get

$$s^2(s+4)^2Y=s^3+6s^2+C.$$

Then

$$Y(s)= \frac{s}{(s+4)^2}+\frac{6}{(s+4)^2}+\frac{C}{s^2(s+4)^2}$$

Upon applying the inverse Laplace transform, we obtain

$$y(t)=e^{-4t}+2te^{-4t}+ \frac{C}{32} [-1+2t+e^{-4t}+2te^{-4t}].$$

This function satisfies the differential equation and the condition $y(0)=1$ for any real number C. This problem does not have a unique solution.

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