An Introduction to Sequence and Series

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Abstract:

This research paper is a general overview of Sequence and Series.A sequence is an ordered list of numbers. The sum of the terms of a sequence is called a series. While some sequences are simply random values, other sequences have a definite pattern that is used to arrive at the sequence's terms. In this paper, we have studied about Progression(s) and their types which includes Arithmetic Progression, Geometric Progression and Harmonic Progression. Further we have studied properties of these progressions. Formulas for finding n^{th} term, selections of n terms, sum of n terms for a given progression and insertion of n number of mean(s) between two given numbers are also studied.

Keywords:

AP; GP; HP; Sequence; Series ; A_m; G_m; H_m; S_n

Introduction

A sequence is a list of numbers. Any time you write numbers in a list format, you are creating a sequence. Something as simple as1, 2, 3, 4, 5, 6, . . .is a sequence.Rather than just listing the numbers, we usually identify it as a sequence with the notationan = 1, 2, 3, 4, 5, 6, . . .Usually there is some type of pattern to a sequence. In the sequence above, you are adding one to each term to get the next term.A term of a sequence is just a number that is in the sequence. Terms can be identified by their location. We note the 1st term in a sequence as a1 and we would call the 5th term in the sequence a5.We described the pattern in the sequence as adding one to each term to get the next term[1]. We can express this as a recursive formula by writing (an = an-1 + 1).This says to get any term in the sequence (an), add one (+1) to the previous term (an-1).

A recursive formula is written in such a way that in order to find any term in a sequence, you must know the previous terms. In other words, to find the 12th term, you would need to know the first 11. There are times when this can be a difficult task and there will be other ways to write sequences. But it is important to know that many sequences are best described using recursive formulas.The simple sequence we have been looking at is called an arithmetic sequence. Any time you are adding the same number to each term to complete the sequence, it is called an arithmetic sequence. The number that is added to each term is called the common difference and denoted with the letter d. So in our example we would say that d = 1. The common difference can be subtracting two consecutive terms. You can subtract any two terms as long as they are consecutive. So we could find d by taking 5 - 4 = 1 or 2 - 1 = 1. Notice that we will always use the term that appears later in the sequence first and then subtract the term that is right in front of it.If we looked at a sequence like bn = 1, 3, 9, 27, 81, 243, . . . this would not fit our definition of an arithmetic sequence. We are not adding the same number to each term. However, notice that we are multiplying each term by the same number (3) each time. When you multiply every term by the same number to get the next term in the sequence, you have a geometric sequence. Geometric sequences can also be written in recursive form. In this case, we would write. Remember that in the language of sequences we are saying, to find any term in the sequence (bn), multiply the previous term (bn-1) by 3.

Just as arithmetic sequences have a common difference, geometric sequences have a common ratio which is denoted with the letter r. The common ratio is found by dividing successive terms in the sequence. So in our geometric sequence example, we could use 9/3 = 3 or 243/81=3 to find that r = 3. As with finding a common difference, when we find a common ratio, we must use the term that appears later in the sequence as our numerator and the number right before it as our denominator[2].

Progression

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula. Those sequences whose terms follow certain pattern are called progression.

1. Arithmetic Progression (AP)

A sequence is called an arithmetic progression if the difference of a term and the previous term is always same, i.e., $a_{n+1} - a_n$ = constant (=d), for each n belongs to N.

The constant difference, generally denoted by d is called the common difference[3].

e.g.

i. 1, 4, 7, 10,.....is an AP whose first term is 1 and common difference is 4 – 1 = 3.

ii. 11, 7, 3, -1,.... is an AP whose first term is 11 and the common difference is 7 – 11 = -4.

1.1 Method to determine whether a Sequence is an AP or not when its n^{th} term is given

i. Obtain a_n

ii. Replace n by $n+1$ in a_n to get a_{n+1}.

iii. Calculate $a_{n+1} - a_n$.

If $a_{n+1} - a_n$ is independent of n, the given sequence is an AP otherwise it is not an AP.

1.2 General Term of an AP

Let a be the first term and d be the common difference of an AP. Then the n^{th} term is a + (n - 1)d.

i.e., T_n= a + (n - 1)d

If l is the last term of a sequence, then

$$l = T_n = a + (n - 1)d$$

1.3 n^{th} term of an AP from the end

Let a be the first term and d be the common difference of an AP having $m$terms. Then, n^{th} term from the end is $(m – n + 1)^{th}$ term from the beginning. So, nth term from the end is

$$T_{m-n+1} = a + (m – n + 1 - 1)d = a + (m - n)d$$

1.4 Selection of terms in an AP

Sometimes we require selecting a certain numbers of terms in AP. The following ways of selecting terms are generally very convenient.

No. of terms	Terms	Common Difference
3	a-d, a, a+d	d
4	a-3d ,a-d, a+d,a+3d	2d
5	a-2d, a-d, a ,a+d ,a+2d	d
6	a-5d, a-3d, a-d, a+d, a+3d. a+5d	2d

1.5 Sum of n terms of an AP

The sum S_n of an AP with first term a and common difference d is

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Or

$$S_n = \frac{n}{2}[a + l]$$

Where, l =last term = a + (n - 1)d

Also $T_n = S_n - S_{n-1}$

1.6 Insertion of n Arithmetic Means

Let $A_1, A_2, A_3, \ldots, A_n$ be n arithmetic means between two quantities a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an AP[4].

Let d be the common difference of this AP, Clearly it contains (n + 2) terms

∴ b = $(n + 2)^{th}$ term

⇨ $b = a + (n + 1)d$

⇨ $d = \frac{b - a}{n + 1}$

In general m^{th} arithmetic mean is given by

$$A_m = a + \frac{m(b-a)}{n+1}$$

By putting m = 1, 2, 3,...., n, we get $A_1, A_2, \ldots, A_n$, which are required arithmetic means.

If $n = 0$, then $A_m = \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$

If there is only one arithmetic mean A between a and$b$, then a, A, b are in AP.

Then, $A = \frac{a+b}{2}$ here $n = 1$

1.7 Properties of Arithmetic Progression

i. If a constant is added or subtracted from each term of an AP, Then the resulting sequence is also an AP with same common difference.

ii. If each term of a given AP is multiplied or divided by a non-zero constant k, then the resulting sequence is also an AP with common difference Kd or $\frac{k}{d}$, where d is the common difference of the given AP.

iii. A sequence is an AP iff its n^{th} term is of the form A_n+B i.e., a linear expression in *n*. The common difference in such a case is *A* i.e., the coefficient of *n*.

iv. In a finite AP the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.i.e., $a_1+a_n = a_2 + a_{n-1} = a_3+a_{n-2}$= ......

v. If the terms of an AP are chosen at regular intervals then they form an AP.

vi. If p^{th} term of an AP is *q* and q^{th} term is *p*, then $T_{p+q} = 0$, $T_r = p + q - r$.

vii. If $S_p = S_q$ for an AP, then $S_{p+q} = 0$

viii. Sum of nA_m's between *a* and*b* is equals to *nA*.

2. Geometric Progression (GP)

A sequence of non-zero numbers is called a Geometric Progression if the ratio of a term and the term proceeding to it is always a constant quantity.

The constant ratio, generally denoted by *r* is called the common ratio of the GP.

e.g., the sequence$\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, -\frac{9}{8}$,..... is a GP with first term $\frac{1}{3}$ and common ratio$(-\frac{1}{2})/(\frac{1}{3})$ = $(-\frac{3}{2})$.

2.1 General term of a GP

The n^{th} term of a GP with first term *a* and common ratio *r* is given by $a_n = ar^{n-1}$

GP can be written as: *a*, *ar*, ar^2,......,ar^{n-1}

or

a, ar, ar^2, ar^3, ar^4,......, ar^{n-1}

according as it is finite or infinite.

2.2 The *n*th term from the end of a finite GP

The n^{th} term from the end of a finite GP consisting of *m* terms isar^{m-n}, where *a* is the first term and *r* is the common ratio of the GP[5].

2.3 Selection of Terms in GP

Sometimes it is required to select a finite number of terms in GP. It is convenient if we select a term in following manner.

No. of terms	Terms	Common ratio
3	$\frac{a}{r}$ a, ar	r
4	$\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3	r^2
5	$\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2	r

2.4 Sum of n terms of a GP

The sum of n terms of a GP with first term 'a' and common ratio 'r' is given by

$S_n = a(\frac{1-r^n}{1-r})$ for $|r|<1$

and $S_n = \frac{a(r^n-1)}{r-1}$ for $|r|>1$

The thing that must be noted is that the above formulae do not hold for *r* = 1. For *r* = 1, the sum of *n* terms of the GP is $S_n = na$.

If number of terms is infinite then the sum of the terms is

S = $\frac{a}{1-r}$ for |r|<1

2.5 Insertion of n Geometric Means

Let G_1, G_2,.....,G_n be *n* geometric means between two quantities *a* and *b*, Then, a, G1, G2,......, Gn, b is a GP.

Clearly, it contains (n + 2) terms. Let *r* be the common ratio of this GP.

∴ b = (n + 2)th term = ar^{r+1}

⇨ r = $\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

In general m^{th} geometric mean is given by

$$G_m = a\left(\frac{b}{a}\right)^{\frac{m}{n+1}}$$

By putting *m* = 1, 2, 3,....., n, we get G_1, G_2,......, G_n. Which are the required geometric means.

If n = $-\frac{1}{2}$, then, $G_m = \frac{a^{n+1} + b^{n+1}}{a^n + b^n}$

If there is only one geometric mean *G* between *a* and*b*, then *a*, *G*, *b* are in GP.

Then, $G_m = (ab)^{\frac{1}{2}}$

2.6 Properties of Geometric Progression

i. If all the terms of a GP be multiplied or divided by a non-zero constant, then it remains a GP with same common ratio[6].

ii. The reciprocals of the terms of a given GP form a GP.

iii. If each term of a GP be raised to same power, the resulting sequence also forms a GP.

iv. In a finite GP the product of terms equidistant from the beginning and the end is always same and is equal to the product of first and last term.

v. Three non-zero numbers *a*, *b* and *c* are in GP, iff $b^2 = ac$.

vi. If the terms of a given GP are chosen at regular intervals, then the new sequence so formed also forms a GP[7].

vii. If a_1, a_2, a_3,....., a_n,.... is a GP of non-zero, non-negative terms, then log a_1, log a_2,......,log a_n,.... is an AP and vice-versa.

viii. If *a*, *b* and *c* are p^{th}, $q^{th}$and r^{th} terms of a GP respectively, then $a^{q-r}.b^{r-p}.c^{p-q} = 1$.

3. Harmonic Progression (HP)

A sequence a_1, a_2,......, a_n of non-zero numbers is called a Harmonic Progression, if the sequence of reciprocal of these numbers i.e., $\frac{1}{a1}, \frac{1}{a2}, \ldots\ldots, \frac{1}{an}$ is an AP.

e.g., the sequence $\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$ is a HP because the sequence 1, 3, 5, 7,...... is an AP.

There is no formula for finding the sum of HP sequence[8,9].

3.1 General term of a HP

If the sequence a_1, a_2, a_3,.....is a HP, then its n^{th} term

$$Tn = \frac{1}{\frac{1}{a1} + (n-1)\left(\frac{1}{a1} - \frac{1}{a2}\right)}$$

3.2 Insertion of *n* Harmonic Means

Let H_1, H_2,......,H_n be *n* harmonic means between two quantities *a* and *b*. Then, *a*, H_1, H_2,.......,H_n, *b* is a HP.

Cleary, it contains (n + 2) terms.

Also, $\frac{1}{a}, \frac{1}{H1}, \frac{1}{H2}, \ldots\ldots, \frac{1}{Hn}, \frac{1}{b}$ is an AP.

Let the common difference of this AP is *d*. Then,

$\frac{1}{b}$ = (n + 2)th term

⇨ $$\frac{1}{b} = \frac{1}{a} + (n + 1)d$$

⇨ $$d = \frac{a-b}{(n+1)ab}$$

In general, m^{th} harmonic mean is given by

$$Hm = \frac{(n+1)ab}{ma + [n-(m-1)]b}$$

for, $m = 1, 2, \ldots, n$.

By putting *m*=1, 2,......,*n*, we get H_1, H_2,......, H_n which are required geometric means.

If *n* = -1, then, Hm = $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$

If there is only one harmonic mean *H* between *a* and*b*, then *a*, *H*, *b* are in HP.

Then, $$H = \frac{2ab}{a+b}$$

3.3 Properties of Harmonic Mean

If the m^{th} term of HP = *n* and n^{th} term = *m*, then $T_{m+n} = \frac{mn}{m+n}$ and $T_{mn} = 1$.

4. Relationship between Arithmetic, Geometric and Harmonic Means

Let A, G and H be the arithmetic, geometric and harmonic means between *a* and *b*, then

i. A >= G >= H

ii. $G^2 = AH$

iii. The equation having *a* and *b* as its roots is $x^2 - 2Ax + G^2 = 0$

iv. If A_1, A_2 be two Am's; G_1, G_2 be two Gm's and H_1, H_2 be two Hm's between two numbers *a* and *b*, then, $\frac{G1G2}{H1H2} = \frac{A1 + A2}{H1 + H2}$.

5. Sum of *n* terms of special series

i. The sum of first *n* natural numbers

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

ii. The sum of squares of the first *n* natural numbers

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

iii. The sum of cubes of the first *n* natural numbers

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Conclusion

Any arithmetic progression is completely determined by any two its quantities.A progression is another way of saying sequence, thus a Geometric Progression is also known as a Geometric Sequence.A Geometric Progression is a special sequence defined by the special property that the ratio of two consecutive terms is the same for all the terms in the sequence. Whereas in Arithmetic Progression we talked of difference, here we talk of ratios meaning that when you divide the current term by the previous term the number that you get should be a non zero constant that is the same for all the consecutive pairs of terms in the sequence andAharmonic progression (or harmonic sequence) is a progression formed by taking the reciprocals of an arithmetic progression.

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